Jacobian trees and their applications

Doktori értekezés

Sigray István

Matematika Doktori Iskola

vezetője: Laczkovich Miklós akadémikus

Elméleti Matematika Doktori Program

vezetője: Szenthe János a matematika tudomány doktora

Témavezető:

Némethi András

Tudományos tanácsadó, a matematika tudomány doktora

Eötvös Loránd Tudományegyetem

2008

Jacobian trees and their applications

PhD Thesis

István Sigray

Mathematics PhD School of Eötvös Loránd University

Head: prof. Miklós Laczkovich, member of the Hungarian academy of Sciences

Pure Matematics PhD Program

Head: prof. János Szenthe D.Sc.

Advisor:
András Némethi D.Sc.

Eötvös Loránd University

Faculty of Sciences

2008

Contents

1 Introduction

In this section we formulate the Jacobian conjecture, and outline the plan of our efforts to achieve some results about it.

Notation 1.1. *We use the symbol* $\ominus$ *for a non-determined constant in* $\mathbb{C}^*$. *There are some arithmetical rules we will use:*

(i) $\ominus^2 = \ominus$.

(ii) For any $c \in \mathbb{C}^*$, $c\ominus = \ominus$.

Notation 1.2. *Let* $f_1, \ldots, f_n : \mathbb{C}^n \to \mathbb{C}$ *be polynomials. We say that* $(f_1, \ldots, f_n)$ *admits the Jacobian condition if*

$$J(f_1, \ldots, f_n) = \ominus. \tag{1}$$

Elementary calculation shows that if the polynomial mapping $F = (f_1, \ldots, f_n)$ is invertible, then $(f_1, \ldots, f_n)$ must be a Jacobian pair. The famous Jacobian conjecture, essentially asked by Keller [Ke], asserts that each polynomial mapping admitting the Jacobian condition must be invertible. Aside from the trivial case $n = 1$ it remains an open problem for all $n \geq 2$.

Keller, who curiously considered F with integer coefficients, verified the conjecture in the birational case, i.e. when F has an inverse formed of rational functions.

The analogue in characteristic $p > 0$ is false, already for $n = 1$, $F(x) := x + x^p$. The analytic analogue is likewise false, for example with $n = 2$, $f_1(x, y) := e^x$, $f_2(x, y) := e^{-x}y$. There is even an entire $F : \mathbb{C}^2 \to \mathbb{C}^2$ with $J(F) = \ominus$ such that F is injective, but $F(\mathbb{C}^2)$ misses a nonempty open set (cf. [B-M]).

If, however, $F : \mathbb{C}^n \to \mathbb{C}^n$ is polynomial mapping admitting the Jacobian condition, and F is injective, then F is invertible. Thus the Jacobian Conjecture must depend on properties specific to polynomials in characteristic zero.

The so called "strong real Jacobian conjecture" was disproved in 1994 when Pintchuk gave an example of a non-injective polynomial mapping from $\mathbb{R}^2$ into itself whose Jacobian determinant is everywhere positive on $\mathbb{R}^2$ (cf. [P]).

In this thesis we will consider the Jacobian conjecture in the case $n = 2$, which is the so called plane Jacobian conjecture. For simplicity we use a slightly different notation.

Notation 1.3. *Let $f,\ g : \mathbb{C}^2 \to \mathbb{C}$ be polynomials. We say that (f,g) is an automorphism, if $(f,g) : \mathbb{C}^2 \to \mathbb{C}^2$ is a bijection.*

We say that (f,g) is a Jacobian pair if it admits the Jacobian condition

$$J(f,g) = \ominus. \tag{2}$$

The plane Jacobian conjecture asserts that any Jacobian pair is an automorphism.

Next we list some results proved only for the plane Jacobian conjecture. Notable among these is Moh's proof of the plane Jacobian conjecture when $\max(\deg(f), \deg(g)) < 100$ (cf. [M]).

Abhyankar in [A] proved the conjecture in the Galois case when $\mathbb{C}(x,y)$ is Galois over $\mathbb{C}(f,g)$. Moreover, Abhyankar also proved that the conjecture is equivalent with the property that the curves $f = 0$ and $g = 0$ have only one point at infinity in $\mathbb{P}^2$. He also shows that they have at most two points at infinity, a result also proved by Makar-Limanov [M-L].

Kaliman proved that any Jacobian pair can be composed by a convenient automorphism such that any fiber of the composition is irreducible (cf. [Ka]). Razar, Heitmann and Weber independently showed that the conjecture holds if f is rational (which means that the generic fiber is the punctured sphere), and any fiber of f is irreducible (cf. [He], [L-W] [N-N] and [R]). Lê showed recently that the conjecture is true if f is rational.

Another point of view is to consider the topological degree of the Jacobian pair (f,g). (The topological degree is evidently the degree of the extension $[C(x,y) : C(f,g)]$.) By the above remarks (e.g. by the work of Keller), it is enough to prove that the topological degree, say d, is 1 for any Jacobian pair. Moreover, $d \neq 2$, because in this case the field extension above were Galois. Orevkov proved that $d \neq 3$ cf. [O 2], Orevkov and Domrina proved that $d \neq 4$ cf. [D-O] and [D].

The main result of this thesis is that $d \neq 5$. To achieve this result we will consider the Eggers – Wall tree of the singularities of the fibers $f(x,y) = a$. We will use special decoration on the tree, which will not only depend on the singularity.

Our plan is the following. We consider a possible counterexample (f,g) of the Jacobian conjecture, and consider some properties of (f,g).

Notation 1.4. *Assume (f,g) is a Jacobian pair. Denote the fibers of f by $R_a := f^{-1}(a)$. We regard R_a as a finite union of Riemann surfaces, and we denote the smooth compactification of R_a by $\overline{R}_a$.*

Set $f := f(x,y)$, $g := g(x,y)$. On the compactified fibers R_a one defines the meromorphic functions x, y and g by natural restriction. Evidently,

$\overline{R}_a \setminus R_a$ is a finite set containing the poles of x and the poles of y. By (2) the meromorphic function g is non-constant. Moreover g has another remarkable property stated below:

Notation 1.5. *Set $P \in \overline{R}_a$. Denote the multiplicity of g at P by $\Lambda(P)$.*

Statement 1.1. *Assume (f, g) is a Jacobian pair. Then*

(i) For any $P \in R_a$ one has $\Lambda(P) = 1$.

(ii) For any $P \in R_a$ one has $g(P) \in \mathbb{C}$.

It is worth to mention, that the fibers corresponding to a Jacobian pair are always smooth. Moreover, we obtain more interesting properties about the fibers, if we compactify them.

This statement shows that both the poles and the critical points of the non-constant meromorphic function g form a subset of the finite set $\overline{R}_a \setminus R_a$. Our plan is to obtain a more detailed description of the poles and the critical points of g, which leads to some geometrical properties of the possible counterexamples of the Jacobian conjecture.

In Section 2 we introduce a normal form for (f, g). This normal form goes back to Abhyankar, and its existence is guaranteed by some results of Abhyankar. It also implies that the fibers have two points at infinity. Moreover, this normal form will can be applied by the Eggers – Wall tree.

In Section 3 we introduce the concept of Eggers – Wall tree, and introduce some requiring notations.

In Section 4 we extend the fundamental theorem of Abhyankar of [A] (cf. Proposition 4.2 and Proposition 4.3). We introduce a partial ordering naturally defined by Propositions 4.2 and 4.3. We essentially prove that any two points of the Eggers Wall tree close enough can be compared by the above partial ordering (Proposition 4.4 and Proposition 4.5).

Section 5 presents a formula how one can calculate the topological degree from the data of the Eggers – Wall tree (Proposition 5.5 and Proposition 5.8).

In Section 6 we characterize the partial ordering defined in Section 4. We introduce the symbols $\searrow$ and $\nearrow$. There exists a strict connection between these symbols and the partial ordering (cf. Proposition 6.3). In the end of the section we characterize the structure of the symbols $\searrow$ and $\nearrow$ (cf. Proposition 6.6 and Proposition 6.7).

In Section 7 we give a formula for the branching data corresponding to the finite critical values of g. We also give an upper bound for the number of critical points with finite critical values for fixed topological degree (cf. Corollary 7.1).

In Section 8 we prove some number-theoretical properties of Jacobian pairs. The fundamental result of this section is Proposition 8.1 which shows a relation between the number-theoretic properties of neighbouring vertices of the Eggers – Wall tree.

In Section 9 we define a decoration on the vertices of the Eggers – Wall tree. This decoration helps to prove that there does not exist a Jacobian pair with topological degrees 2, 3, 4 and 5 (cf. Theorem 9.1).

Although the present method eliminates the new case when the topological degree is 5, it is not strong enough to solve completely the Jacobian conjecture. It is not only true that in the higher degree cases the arithmetical computations are more involved, but one can also create a decorated Eggers Wall tree (corresponding to topological degree 9) which would not contradict to any of the restrictions considered in this thesis. It is worth to mention that the corresponding polynomials f and g satisfy $\deg(f) = 48$ and $\deg(g) = 64$ which was one of the four exceptional cases in Moh's paper [M].

On the other hand, the method can be compared with the method of Moh, and we believe that it is even a sharper filter.

Moreover, we think that analyzing the topological degree, we set a more geometric picture then via the degrees of the polynomials f and g.

2 Normal forms

In this section we give a normal form of the possible counterexamples. The results of the sections are simple corollaries of Abhyankar's results [A].

Notation 2.1. *Suppose (f, g) is a Jacobian pair. Define the equivalence $\sim$ by $(f, g) \sim (f^*, g^*)$, if there are $\mathfrak{K}$ and $\mathfrak{L}$ automorphisms with the property*

$$(f^*, g^*) = \mathfrak{K} \circ (f, g) \circ \mathfrak{L}.$$

Assume (f, g) is chosen from its $\sim$ class so that $(\deg f, \deg g)$ *is minimal in the lexicographic order. Then we say that (f, g) is an almost normalized Jacobian pair.*

We say that an almost normalized Jacobian pair (f, g) is an almost normalized counterexample of the Jacobian conjecture if it is not an automorphism.

Notation 2.2. *Define $\mathbb{Q}^+ := \mathbb{Q} \cap [0, \infty)$.*

The almost normalized counterexamples have some remarkable properties.

Lemma 2.1. *Suppose $(\widetilde{f}, \widetilde{g})$ is an almost normalized counterexample of the Jacobian conjecture. Then there exists an almost normalized counterexample of the Jacobian conjecture in the form $(f, g) = (\widetilde{f} \circ A, \widetilde{g} \circ A)$, (where $A : \mathbb{C}^2 \to \mathbb{C}^2$ is a non-degenerated linear mapping), whose Newton polygons N_f and N_g have the following properties:*

(i) There exist $(k_f, l_f) \in N_f$ and $(k_g, l_g) \in N_g$ with $k_f, l_f, k_g, l_g > 0$, and N_f is a part of the rectangle with vertices $(0,0), (0, l_f), (k_f, l_f)$, $(k_f, 0)$, and N_g is a part of the rectangle with vertices $(0,0), (0, l_g), (k_g, l_g)$, $(k_g, 0)$.

(ii) $\frac{k_f}{k_g} = \frac{l_f}{l_g}$;

(iii) $k_f < k_g$, $l_f < l_g$ $l_f \leq k_f$ and $l_g \leq k_g$;

(iv) $\frac{k_g}{k_f} \notin \mathbb{N}^*$;

Proof. We use the notations of Abhyankar [A]. Set $\mathrm{w} = (w_1, w_2) \in \mathbb{R}^2$. The w-degree of the monomial $\ominus x^i y^j$ is defined by $\deg_{\mathrm{w}}(\ominus x^i y^j) := w_1 i + w_2 j$.

Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be a non-zero polynomial. Let $\deg_{\mathrm{w}}(H)$ be the maximal w degree of the non-zero monomials of H, and define H^+_{w} by the sum of the non-zero monomials of H with maximal w degree.

By [A, Theorem 18.13]

$$\widetilde{f}^+_{1,1}(x, y) = \ominus (a_1 x + b_1 y)^{k_f} (a_2 x + b_2 y)^{l_f},$$

for some constants $a_1, a_2, b_1, b_2 \in \mathbb{C}$ and k_f, $l_f \in \mathbb{N}$. By symmetry we may assume $l_f \leq k_f$. By [A, Theorem 19.2] $k_f \neq 0$ and $l_f \neq 0$ and $a_1 b_2 - a_2 b_1 \neq 0$. Set

$$A := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1}.$$

Set $(f, g) := (\widetilde{f} \circ A, \widetilde{g} \circ A)$.

By the definition of A we have $f^+_{(1,1)}(x, y) = \ominus x^{k_f} y^{l_f}$. By the definition, (k_f, l_f) is a vertex of N_f. We state that N_f is a part of the rectangle with vertices $(0,0), (0, l_f), (k_f, l_f)$, $(k_f, 0)$.

If there were, however a point $(i, j) \in N_f$ with either $i > k_f$ or $j > l_f$, then we also could assume that $((i, j), (k_f, l_f))$ is an edge of N_f. Moreover, since $f^+_{(1,1)}(x, y) = \ominus x^{k_f} y^{l_f}$, we have $i + j < k_f + l_f$.

Suppose for example $i > k_f$. Then, evidently $j < l_f$. Set w$=(l_f - j, i - k_f)$. By [A, Theorem 18.13] (using $l_f - j > i - k_f$ and the normalization property of f) we obtain that f^+_{w} is a monomial. This is, however a contradiction, since, by the definition of the Newton polygon, both $\ominus x^{k_f} y^{l_f}$ and $\ominus x^i y^j$ are non-zero terms of f^+_{w}.

By [A Lemma 18.2], for any $\mathrm{w} \in \mathbb{R}^2$, $J(f_{\mathrm{w}}^+, g_{\mathrm{w}}^+) = c \in \mathbb{C}$. Set $\mathrm{w} = (w_1, w_2)$ with w_1, $w_2 > 0$. By the property of N_f just proved above, we have $f_{\mathrm{w}}^+ = \ominus x^{k_f} y^{l_f}$. Therefore, $J(f_{\mathrm{w}}^+, g_{\mathrm{w}}^+) = \ominus x^{k_f} y^{l_f - 1} \frac{\partial g_{\mathrm{w}}^+}{\partial x} + \ominus x^{k_f - 1} y^{l_f} \frac{\partial g_{\mathrm{w}}^+}{\partial y} \neq \ominus$, (because $k_f, l_f > 0$), so $J(f_{\mathrm{w}}^+, g_{\mathrm{w}}^+) = 0$. By [A, Proposition 17.4], $g_{\mathrm{w}}^+ = \ominus (f_{\mathrm{w}}^+)^\alpha = \ominus (x^{k_f} y^{l_f})^\alpha$ for some $\alpha \in \mathbb{Q}^+$. Since w_1, $w_2 > 0$, $\deg(\ominus(x^{k_f} y^{l_f - 1})^\alpha) < \deg(\ominus(x^{k_f} y^{l_f - 1})^\beta)$ for any $0 \leq \alpha < \beta$, we obtain that α does not depend on w. Set $(k_g, l_g) := \alpha(k_f, l_f)$. Since $\ominus x^{k_g} y^{l_g}$ is a monomial of g, one obtains that k_g, $l_g \in \mathbb{N}$. Moreover, $\alpha \neq 0$ since otherwise g would be a constant polynomial which evidently contradicts (2). This shows k_g, $l_g > 0$ and (ii).

The same way, as above, we can prove that the property $g_{\mathrm{w}}^+ = \ominus x^{k_g} y^{l_g}$ for any $\mathrm{w} = (w_1, w_2)$ with w_1, $w_2 > 0$ implies that N_g is a part of the rectangle with vertices $(0,0), (0, l_g), (k_g, l_g)$, $(k_g, 0)$. (In fact these two properties are equivalent.) This shows the remaining part of (i).

Notice that $l_f \leq k_f$, and (ii) implies $l_g \leq k_g$ as well. Moreover, by the normalization property, $(\deg(f), \deg(g)) \leq (\deg(g), \deg(f))$ (in the lexicographic order), and (ii) implies $k_f \leq k_g$ and $l_f \leq l_g$ too. Since by (ii) $k_f = k_g$ if and only if $l_f = l_g$, in order to finish the proof of (iii), it is enough to prove (iv).

Assume, however, that $\frac{k_g}{k_f} = \alpha \in \mathbb{N}^*$. By (ii) and (i), $g_{(1,1)}^+ = c(f_{(1,1)}^+)^\alpha$, for some $c \in \mathbb{C}^*$. Set $g^* = g - cf^\alpha$. By its definition, $(f, g^*) \sim (f, g)$ and $\deg(g^*) < \deg(g)$, which contradicts the normalization property of (f, g).

Notation 2.3. *Let (f, g) be an almost normalized Jacobian counterexample of the Jacobian conjecture admitting properties (i), (ii) (iii) and (iv) of Lemma 2.1. Then we say that (f, g) is a normalized counterexample of the Jacobian conjecture.*

From now on we only deal with the possible normalized counterexamples of the Jacobian conjecture.

Remark: Later on we will prove that $l_f < k_f$ (and so $l_g < k_g$), (cf. Theorem 6.1) but that proof needs a slightly deeper analysis.

Notation 2.4. *Let (f, g) be a normalized counterexample of the Jacobian conjecture. Define α, $\beta \in \mathbb{N}^*$ by*

(i) $\frac{\alpha}{\beta} = \frac{k_f}{k_g}$.

(ii) $\gcd.(\alpha, \beta) = 1$.

We call the pair (α, β) the "type of the counterexample" (f, g).

Statement 2.1. *By (iv) of Lemma 2.1, any normalized counterexample the type (α, β) satisfies $\alpha \neq 1$ and $\beta \neq 1$.*

3 The Eggers – Wall tree

Our plan is to analyze the Puiseux series around the x and y poles of R_a. The normalized counterexamples of the Jacobian conjecture have an important property.

Statement 3.1. *Set $P \in \overline{R_a} \setminus R_a$. Then exactly one of the following two properties holds*

(i) $x(P) = \infty$ and $y(P) \in \mathbb{C}$, moreover

$$y = \sum_{j=0}^{\infty} c_j x^{-j/\kappa} \tag{3}$$

in a small neighbourhood of P, where κ is the multiplicity of x at P.

(ii) $y(P) = \infty$ and $x(P) \in \mathbb{C}$, moreover

$$x = \sum_{j=0}^{\infty} c_j y^{-j/\kappa} \tag{4}$$

in a small neighbourhood of P, where κ is the multiplicity of y at P.

The definitions below are essentially due to Eggers cf. [Egg].

Definition 3.1. *Set $P \in \overline{R}_a \setminus R_a$. Assume that the Puiseux series has the form (3) (or (4) resp.). Set $e_0 := \kappa$ and define inductively*

(i) $\beta_i := \min\{j : \ c_j \neq 0 \ \text{ and } \ j \ \text{ is not divisible by } \ e_{i-1}\}$.

(ii) $e_i := \gcd.(e_{i-1}, \beta_i)$.

After finitely many steps we obtain $e_m = 1$ and the procedure comes to an end. Set $\alpha_{j,P} := \beta_j/\kappa$. The set $(\kappa, \beta_1, \ldots, \beta_m)$ is called the Puiseux characteristics of P. For technical reasons, we use the notation $\alpha_0 = 0$.

Definition 3.2. *Let $\mathcal{P}, \mathcal{P}^*$ be two different Puiseux series either in the form*

$$y = \sum_{j \in \mathbb{Q}^+} c_j x^{-j} \quad \text{and} \quad y = \sum_{j \in \mathbb{Q}^+} c_j^* x^{-j}$$

or in the form

$$x = \sum_{j \in \mathbb{Q}^+} c_j y^{-j} \quad \text{and} \quad x = \sum_{j \in \mathbb{Q}^+}^{\infty} c_j^* y^{-j}.$$

Then define $\mathcal{O}(\mathcal{P}, \mathcal{P}^*) := \min\{j : \ c_j \neq c_j^*\}$.

Set $P \neq P^* \in \overline{R}_a \setminus R_a$. Define

$$\mathcal{O}(P, P^*) := \max\{\mathcal{O}(\mathcal{P}, \mathcal{P}^*) : \mathcal{P} \ \text{ corresponds to } \ P \ \text{ and } \ \mathcal{P}^* \ \text{ corresponds to } \ P^*\}.$$

Otherwise let $\mathcal{O}(P, P^*) = -1$.

Statement 3.2. *There exits a map, say* Ω*, from the finite set* $\overline{R}_a \setminus R_a$ *to the set of formal series in the form*

$$y = \sum_{j \in \mathbb{Q}^+} c_j x^{-j} \quad or \quad x = \sum_{j \in \mathbb{Q}^+} c_j y^{-j},$$

such that

(i) For any $P \in \overline{R}_a \setminus R_a$*,* $\Omega(P)$ *is the Puiseux series of* P*.*

(ii) For any $P, Q \in \overline{R}_a \setminus R_a$*,* $\mathcal{O}(P, Q) = \mathcal{O}(\Omega(P), \Omega(Q))$*.*

Finally define the Eggers – Wall tree.

Definition 3.3. *Define the equivalence* $\sim$ *on* $\overline{R}_a \setminus R_a \times [0, \infty]$ *by* $(P, u) \sim (P^*, u^*)$ *if*

(i) $u = u^*$*.*

(ii) $u \leq \mathcal{O}(P, P^*)$*.*

The Eggers – Wall tree is $T_a^* = ((\overline{R}_a \setminus R_a) \times [0, \infty])^\sim$*.*

Notation 3.1. *Set* $P \in \overline{R}_a \setminus R_a$*. Define the natural embedding* $I_P : [0, \infty] \to T_a^*$ *by* $I_P(u) = (P, u)^\sim$*.*

Define the natural projection $\pi : T_a^* \to [0, \infty]$ *by* $\pi((P, u)^\sim) = u$*, which mapping is well-defined.*

Notation 3.2. *Set*

$S_y^* := \{P \in \overline{R}_a \backslash R_a : \ P \ \ \text{admits a Puiseux series in the form (3) for some} \ \ \kappa \in \mathbb{N}^*\}$

$S_x^* := \{P \in \overline{R}_a \backslash R_a : \ P \ \ \text{admits a Puiseux series in the form (4) for some} \ \ \kappa \in \mathbb{N}^*\}$

Assume $P \in \overline{R}_a \setminus R_a$ *admits a Puiseux series in the form (3) for some* $\kappa \in \mathbb{N}^*$*. Define* $(0, y) := (P, 0)^\sim$*. Assume* $P \in \overline{R}_a \setminus R_a$ *admits a Puiseux series in the form (4) for some* $\kappa \in \mathbb{N}^*$*. Define* $(0, x) := (P, 0)^\sim$*.*

Statement 3.3. T_a^* *has two components,* $T_{a,x}^* := S_{a,x}^* \times [0, \infty]^\sim$ *and* $T_{a,y}^* := S_{a,y}^* \times [0, \infty]^\sim$ *.*

Definition 3.4. *Define*

$$V_{1,a} := \{I_P(u) : \ u = \alpha_{j,P}, \ \ \text{for some} \ \ j \ \ \text{and} \ \ P \in \overline{R}_a \setminus R_a\},$$

$$V_{2,a} := \{I_P(u) : \ u = \mathcal{O}(P, P^*), \ P, P^* \in \overline{R}_a \setminus R_a\}$$

and $V_a := V_{1,a} \cup V_{2,a} \cup (0, x) \cup (0, y)$*.* V_a *is called the set of vertices of the Eggers – Wall tree.*

The topology of T_a^* naturally defines the set of edges. We formalize, how we obtain neighbouring vertices.

Notation 3.3. *Set $\mathcal{F} \in T_a \setminus \{(0,x),(0,y)\}$, $\mathcal{F} = I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$ $u \in [0,\infty]$. Assume $u^* < u$, $u^* \in \mathbb{Q}$ satisfies*

(i) $I_P(u^*) \in V_a$.

(ii) $I_P((u^*,u)) \cap V_a = \emptyset$.

Then define $\mathcal{F}^\circ := I_P(u^*)$, which is independent of the choice of P.

Moreover, if $\mathcal{F} \in V_a$, then the edge between the vertices $\mathcal{F}$ and $\mathcal{F}^\circ$ is $e_{\mathcal{F},\mathcal{F}^\circ} := I_P([u,u^*])$. The set of edges is exactly

$$E_a := \{e_{\mathcal{F},\mathcal{F}^\circ} :\ \mathcal{F} \in V_a \setminus \{(0,x),(0,y)\}\}.$$

We introduce a different decoration to the set V_a.

Notation 3.4. *Set $\mathcal{F} \in V_a$. Define $\nu_\mathcal{F} \in \mathbb{N}^*$ in the following way. In the case $\mathcal{F} \notin V_{1,a}$, then let $\nu_\mathcal{F} := 1$.*

Set $\mathcal{F} \in V_{1,a}$. Then $\mathcal{F} = I_P(\alpha_j)$ for some $P \in \overline{R}_a \setminus R_a$, where $(\kappa,\beta_1,\dots\beta_m)$ is the Puiseux characteristics of P, and $\alpha_j = \frac{\beta_j}{\kappa}$. Set $\nu_\mathcal{F} := \frac{e_{j-1}}{e_j}$.

Notation 3.5. *Set $\mathcal{F} \in V_a$. Assume $\mathcal{F} = I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$, $u \in \mathbb{Q}^+$, and assume that the Puiseux characteristics of P is $(\kappa,\beta_1,\dots,\beta_s)$. Set $\kappa_\mathcal{F} := \frac{\kappa}{e_j}$, where $\alpha_j \le u < \alpha_{j+1}$.*

Notation 3.6. *Define $T_a := T_a^* \cap \pi^{-1}(\mathbb{Q})$, $T_{a,y} := T_{a,y} \cap T_a$, $T_{a,x} := T_{a,x} \cap T_a$*

Statement 3.4. *Assume $\mathcal{F} \in V_a$. Then $\mathcal{F} \in T_a$.*

Notation 3.7. *Let $\kappa \in \mathbb{N}^*$ be such that any $P \in \overline{R}_a \setminus R_a$ admits a Puiseux series either in the form (3) or (4). Then we say that κ is suitable.*

Notation 3.8. *Let $\kappa \in \mathbb{N}^*$ be suitable. Chose $\mathcal{F} \in T_a$ such that $\kappa\pi(\mathcal{F}) \in \mathbb{N}^*$. Then, evidently $\mathcal{F} = I_P(\pi(\mathcal{F}))$ for some $P \in \overline{R}_a \setminus R_a$. Define $\mathcal{F}'$ by $\mathcal{F}' := I_P(\pi(\mathcal{F}) - \frac{1}{\kappa})$, which is independent of the choice of P.*

Set $\mathcal{F} \in T_a$. Assume that $n := \kappa\pi(\mathcal{F}) \in \mathbb{N}$. Assume that for $c \in \mathbb{C}$ there exists $P \in \overline{R}_a \setminus R_a$ such that

(i) $\mathcal{F} = I_P(\pi(\mathcal{F}))$.

(ii) The Puiseux series $\Omega(P)$ is in the form (3) or (4) with $c_n = c$.

Then define $\mathcal{F} * c := I_P(\frac{n+1}{\kappa})$. It is easy to check, that $\mathcal{F} * c$ does not depend on the choice of P.

Remark: Actually $\mathcal{F}'$ and $\mathcal{F} * c$ also depends on κ, but for simplicity we don't denote this dependence by the more correct but more complicated $\mathcal{F} *_\kappa c$ or $\mathcal{F}'_\kappa$.

Statement 3.5. *Set $\kappa \in \mathbb{N}^*$, $\mathcal{F} \in T_a$ and $c \in \mathbb{C}$. Assume that $\mathcal{F} * c$ exists. Then $(\mathcal{F} * c)' = \mathcal{F}$.*

Statement 3.6. *Let $\kappa \in \mathbb{N}^*$ be suitable, $\mathcal{F} \in T_a$ with $\kappa\pi(\mathcal{F}) \in \mathbb{N}^*$. Then there exists $c \in \mathbb{C}$ such that $\mathcal{F} = \mathcal{F}' * c$.*

Notation 3.9. *Let $\mathcal{F} \in T_{y,a}$ and $h(x,y)$ be a polynomial. Assume that $\mathcal{F} = I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$ and $u \in \mathbb{Q}^+$ and $\Omega(\mathcal{F})$ is $\sum_{j\in\mathbb{Q}} c_j x^- j$. Define*

$$\eta_{\mathcal{F}} := x^{\pi(\mathcal{F})}(y - \sum_{j\in\mathbb{Q},\ j<\pi(\mathcal{F})} c_j x^{-j}).$$

Define the expression $h^{\mathcal{F}}(x,\eta)$ by $h(x,y) = h^{\mathcal{F}}(x,\eta_{\mathcal{F}})$. The same way can be defined the expression $h^{\mathcal{F}}(\eta,y)$ in the case $\mathcal{F} \in T_{x,a}$, $\kappa\pi(\mathcal{F}) \in \mathbb{N}$.

Statement 3.7. *Let $\kappa \in \mathbb{N}^*$ be suitable, $h(x,y)$ be a polynomial, $\mathcal{F} \in T_{y,a}$. Then*

$$h^{\mathcal{F}}(x,\eta) = \sum_{j=-\infty}^{\infty} x^{j/\kappa} p_j(\eta), \tag{5}$$

where p_j is always a polynomial, and except of finitely many j, $p_j \equiv 0$.

Set $\mathcal{F} \in T_{x,a}$ with $\kappa\pi(\mathcal{F}) \in \mathbb{N}$. Then

$$h^{\mathcal{F}}(\eta,y) = \sum_{j=-\infty}^{\infty} y^{j/\kappa} p_j(\eta), \tag{6}$$

where p_j is always a polynomial, and except of finitely many j, $p_j \equiv 0$.

Notation 3.10. *Set $\mathcal{F} \in T_{y,a}$ with $\pi(\mathcal{F}) \in \mathbb{Q}^+$. Assume $h^{\mathcal{F}}(x,\eta)$ has the form (5). Then define $h^+_{\mathcal{F}}(\xi,\eta) := \xi^{j/\kappa} p_j(\eta)$, where j is the largest index for which $p_j \not\equiv 0$.*

Set $\mathcal{F} \in T_{x,a}$ with $\pi(\mathcal{F}) \in \mathbb{Q}^+$. Assume $h^{\mathcal{F}}(\eta,y)$ has the form (6). Then define $h^+_{\mathcal{F}}(\xi,\eta) := \xi^{j/\kappa} p_j(\eta)$, where j is the largest index for which $p_j \not\equiv 0$.

In both cases define $d_{h,\mathcal{F}} := \frac{j}{\kappa}$ and $p_{h,\mathcal{F}} := p_j$.

Notation 3.11. *Set $\mathcal{F} \in V_a$, $\mathcal{F} = I_P(u)$. Let $v \in \mathbb{Q}^+$ be the maximal element of $I_P([0,u]) \cap V_{1,a}$. Assume $I_P(v) = I_Q(v)$ for some $Q \in \overline{R}_a \setminus R_a$ such that the Puiseux characteristics of Q is $(\kappa, \beta_1, \ldots, \beta_m)$, and $v = \alpha_j$ for some $0 \le j \le m$.*

Let $h(x, y)$ be a polynomial. Define

$$D_{h,\mathcal{F}} := \kappa_{\mathcal{F}} d_{h,\mathcal{F}}.$$

Define $D_{\mathcal{F}} := D_{f,\mathcal{F}}$.

Statement 3.8. *For any polynomial $h(x, y)$, and for any $\mathcal{F} \in V_a$, one has $D_{h,\mathcal{F}} \in \mathbb{Z}$.*

Notation 3.12. *Set $p(z) = \ominus \prod_{j=1}^{s} (z - z_j)^{\alpha_j}$, $z_j \in \mathbb{C}$ are pairwise different, $\alpha_j \in \mathbb{N}$. Set $z \in \mathbb{C}$. Define*

$$\mathrm{mult}(p, z) = \begin{cases} \alpha_j & \text{if } z = z_j \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *Let $h(x, y)$ be a squarefree polynomial. Let R_h be the finite set of Riemann surfaces of the fiber $h(x, y) = 0$. Let R_h^* be the smooth compactification of R_h. By natural restriction we can define the meromorphic functions x and y.*

Assume that $\kappa \in \mathbb{N}^*$ is a multiple of any order of x-pole.

Let $c_0, c_1 \ldots$ be a sequence of complex numbers. Define the sequence of expressions η_n by the following way

$$\eta_n := x^{n/\kappa} (y - \sum_{j=0}^{n-1} c_j x^{-j}).$$

Define h_n by $h_n(x, \eta_n) := h(x, y)$. Then, as we saw by the propoerties of $h^{\mathcal{F}}$, $h_n(x, \eta) = \sum_{j=m}^{d} x^{j/\kappa} p_j(\eta)$, for some polynomials p_j, where $p_m \neq 0$ and $p_d \neq 0$.

(∗) $\deg(p_d)$ *is the number of the Puiseux series of the fiber $h(x, y) = 0$ in the form*

$$y = \sum_{j=0}^{n-1} c_j x^{-j/\kappa} + \sum_{j=n}^{\infty} d_j x^{-j/\kappa}.$$

(∗∗) $\mathrm{mult}(p_d, c_n)$ *is the number of the Puiseux series of the fiber $h(x, y) = 0$ in the form*

$$y = \sum_{j=0}^{n} c_j x^{-j/\kappa} + \sum_{j=n+1}^{\infty} d_j x^{-j/\kappa}.$$

A similar statement is true for the Puiseux series of the fiber $h(x, y) = 0$ in the form (4).

Proof. Let R^* be the smooth closure of the finite set of Riemann surfaces of the fiber $h(t^\kappa, y) = 0$. Then x, y and η_n can be considered as meromorphic functions on R^*. Moreover, there is a one-to-one correspondence between the set of Puiseux series in the form (4) and the set $\{P \in R^* : \ x(P) = \infty \text{ and } y(P) \in \mathbb{C}$.

$$\deg(p_d) = \#\{P \in R^* : \ x(P) = \infty \quad \text{and} \quad \eta_n(P) \in \mathbb{C}\},$$

which implies $(*)$.

$$\text{mult}(p_d, c_n) = \#\{P \in R^* : \ x(P) = \infty \quad \text{and} \quad \eta_n(P) = c_n\},$$

which implies $(**)$. $\square$

Statement 3.9. *Let $h(x, y)$ be a polynomial. Assume $\kappa \in \mathbb{N}^*$ is suitable and has the property of Proposition 3.1. Then for any $\mathcal{F} \in T_a$ with $\kappa\pi(\mathcal{F}) \in \mathbb{N}$, and for any $c \in \mathbb{C}$ such that $\mathcal{F} * c$ is defined, we have*

(i) $\text{mult}(p_{h,\mathcal{F}}, c) = \deg(p_{h,\mathcal{F}*c})$.

(ii) If $l = \deg(p_{h,\mathcal{F}*c})$ *and* $p_{h,\mathcal{F}}(\eta) = \sum_{j=l}^{m} a_j(\eta - c)^j$ *and* $p_{\mathcal{F}*c} = \sum_{j=0}^{l} b_j\eta^j$, *then* $a_l = b_l$.

(iii) $d_{h,\mathcal{F}*c} = d_{h,\mathcal{F}} - \text{mult}(p_{h,\mathcal{F}}, c)/\kappa$.

Proof. Set $h := h_1 \dots h_k$, where $h_1, \dots h_k$ are squarefree polynomials. Since $h^+_{\mathcal{F}} = (h_1)_{\mathcal{F}} \dots (h_k)_{\mathcal{F}}$, it is enough to prove our statement for squarefree polynomials.

Property (i) follows from Proposition3.1. Let us prove (ii) and (iii). Assume $\mathcal{F} \in T_{y,a}$. Set $\mathcal{G} := \mathcal{F} * c$,

$$h^{\mathcal{F}}(x, \eta) = \sum_{j=k}^{n} x^{j/\kappa} p_j(\eta),$$

where $d_{\mathcal{F}} = \frac{n}{\kappa}$, $p_n = p_{h,\mathcal{F}}$. Since $\eta_{\mathcal{G}} = x^{1/\kappa}(\eta_{\mathcal{F}} - c)$, we obtain

$$h^{\mathcal{G}}(x, \eta) = \sum_{j=k}^{l} x^{j/\kappa} p_j(x^{-\kappa}\eta + c).$$

The expression above contains the term $a_l x^{(n-l)\kappa}\eta^l$, moreover, for any other term $ax^s\eta^l$ one has $s < \frac{n-l}{\kappa} = d - \frac{l}{\kappa}$. Since $a_l \neq 0$, by (i) we obtain (ii) and (iii). $\square$

Statement 3.10. *Set* $P \in \overline{R}_a \setminus R_a$*. Define the functions* $\omega, \omega^* : \mathbb{Q} \cap [0, \infty] \to \mathbb{Q}$ *by* $\omega(u) := d_{h,I(u)}$ *and* $\omega^*(u) := \deg(p_{h,I(u)})$*. Then we have the following properties for* ω *and* ω^**.*

(i) ω *and* ω^* *are monotone decreasing.*

(ii) $\omega(u)$ *is continuous.*

(iii) Except of finitely many rational points, the graph of $\omega(u)$ *is locally on a line with slope* $-\omega^*(u)$*.*

This has the following corollary

Statement 3.11. *Let* $h(x, y)$ *be a polynomial,* $\mathcal{F} = \mathcal{F}' * c \in T_a$*. Then*

(i) $\mathrm{mult}(p_{h,\mathcal{F}'}, c) \geq \deg(p_{h,\mathcal{F}})$*.*

(ii) $d_{h,\mathcal{F}} - \mathrm{mult}(p_{h,\mathcal{F}'}, c)/\kappa \leq d_{h,\mathcal{F}} \leq d_{h,\mathcal{F}'} - \deg(p_{h,\mathcal{F}})/\kappa$*.*

Moreover, if in one of the above inequalities equality holds, then each inequality is actually an equality.

Notation 3.13. *Set* $\mathcal{F} \in T_a$ *with* $\pi(\mathcal{F}) \in \mathbb{Q}$ *define* $p_{\mathcal{F}} := p_{f,\mathcal{F}}$ *and* $d_{\mathcal{F}} := d_{f,\mathcal{F}}$*.*

Statement 3.12. *Let* $h(x, y)$ *be a polynomial. Then* $h^+_{(0,y)} = h^+_{(0,1)}$ *and* $h^+_{(0,y)} = h^+_{(0,1)}$*. As consequence we obtain* $d_{h,(0,y)} \geq \deg(p_{h,(0,x)})$ *and* $d_{h,(0,x)} \geq \deg(p_{h,(0,y)})$*.*

Statement 3.13. *For any* $P \in \overline{R}_a \setminus R_a$ *there exists a unique* $u \in \mathbb{Q}$ *such* $d_{I(u)} = 0$*.*

Notation 3.14. *Define*
$T^+_a := \{\mathcal{F} \in T_a : \ \pi(\mathcal{F}) \in \mathbb{Q} \ \text{ and } \ d_{\mathcal{F}} > 0\}$,
$T^0_a := \{\mathcal{F} \in T_a : \ \pi(\mathcal{F}) \in \mathbb{Q} \ \text{ and } \ d_{\mathcal{F}} = 0\}$
$T^-_a := \{\mathcal{F} \in T_a : \ \pi(\mathcal{F}) \in \mathbb{Q} \ \text{ and } \ d_{\mathcal{F}} < 0\}$.

Statement 3.14. *For any* $a, b \in \mathbb{C}$ *there exists an isomorphism*
$M_{a,b} : (T^+_a \cup T^0_a) \to (T^+_b \cup T^0_b)$ *with the following properties:*

(i) For any $\mathcal{F} \in T^+_a \cup T^0_a$ *one has* $\pi(M_{a,b}(\mathcal{F})) = \pi(\mathcal{F})$*.*

(ii) For any $\mathcal{F} \in T^+_a \cup T^0_a$ *one has* $f^+_{M_{a,b}(\mathcal{F})} = f^+_{\mathcal{F}}$*.*

Proof. Set $\mathcal{F} \in T_a^+ \cup T_a^0$. Set $\mathcal{F} := I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$ and $u \in \mathbb{Q}^+$. Assume that the Puiseux series of $\Omega(P)$ is in the form

$$y = \sum_{j \in \mathbb{Q}^+} c_j x^{-j} \quad \text{or} \quad x = \sum_{j \in \mathbb{Q}^+} c_j y^{-j} \text{ resp.}$$

Assume first $\mathcal{F} \in T_a^+ \cup T_a^0$. Then

$$(f-b)^+_{\mathcal{F}} = \begin{cases} (f-a)^+_{\mathcal{F}} & \text{if } \mathcal{F} \in T_a^+ \\ (f-a)^+_{\mathcal{F}} - (b-a) & \text{if } \mathcal{F} \in T_a^0. \end{cases}$$

Therefore from Proposition 3.1 we obtain that there exists $Q \in \overline{R}_b \setminus R_b$ such that Q admits a Puiseux series

$$y = \sum_{j \in \mathbb{Q}^+} d_j x^{-j} \quad \text{or} \quad x = \sum_{j \in \mathbb{Q}^+} d_j y^{-j} \text{ resp.}$$

with $d_j = c_j$ if $j < u$. Define $M_{a,b}(\mathcal{F}) := I_Q(u) := \mathcal{G}$. Because ${}_Q(u)$ depends only on the constants d_j $j < u$, we obtain that $M_{a,b}$ is well-defined. By its definition (i) evidently holds. Moreover, $\eta_{\mathcal{F}} = \eta_{\mathcal{G}}$, so (ii) also holds.

Since, by the definition $(M_{a,b} \circ M_{b,a})(\mathcal{F}) = \mathcal{F}$, and $(M_{b,a} \circ M_{a,b})(\mathcal{F}) = \mathcal{F}$, $M_{a,b}$ is an isomorphism. $\square$

Statement 3.15. *Set $P \in \overline{R}_a \backslash R_a$ having Puiseux characteristics $(\kappa, \beta_1, \ldots, \beta_m)$, $h(x,y)$ be a polynomial and $u \in \mathbb{Q}$ such that for $\mathcal{F} := I_P(u)$ $p_{\mathcal{F}}$ and $p_{h,\mathcal{F}}$ has no common root. Then we have the following possibilities:*

(i) $d_{h,\mathcal{F}} > 0$. Then $h(P) = 0$.

(ii) $d_{h,\mathcal{F}} = 0$. Then $h(P) \in \mathbb{C}^*$.

(iii) $d_{h,\mathcal{F}} < 0$. Then $h(P) = \infty$.

Moreover, in the cases (i) and (iii), the multiplicity of $h(P)$ at P is $\kappa |d_{h,\mathcal{H}}|$.

From Statements (3.9) and (3.15) we obtain that there is a strict connection between the polynomial $p_{\mathcal{F}}$ and the coefficients of the Puiseux series corresponding to some $P \in \overline{R}_a \setminus R_a$.

Statement 3.16. *Set $\mathcal{F} \in T_a \setminus \{(0,x),(0,y)\}$. Then $\mathcal{F} \in V_{1,a} \cup V_{2,a}$ if and only if $p_{\mathcal{F}}$ has more than one root.*

Set $\mathcal{F} \in V_a$ and let $\nu := \nu_{\mathcal{F}}$. Then there exists a polynomial $\widetilde{p}$ and $l \in \mathbb{N}$ such that $p_{\mathcal{F}}(\eta) = \eta^l \widetilde{p}(\eta^\nu)$.

Proposition 3.2. *Set $\mathcal{F} \in V_a \setminus \{(0,x),(0,y)\}$ and $\mathcal{G} := \mathcal{F}^\circ$. Assume $\mathcal{F} = I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$. Choose some suitable $\kappa \in \mathbb{N}^*$. Then there exists a unique $c \in \mathbb{C}$ such that $\mathcal{G} * c = I_P(v)$ for some $v \in \mathbb{Q}^+$. The number c does not depend on the choice κ. Therefore, in this situation we may use the notation $\mathcal{F} := \mathcal{G} + c$.*

Statement 3.17. *Set $\mathcal{F}, \mathcal{G} \in V_a$. Assume $\mathcal{F} = \mathcal{G} + c$ for some $c \in \mathbb{C}$. Then we have the following properties:*

(i) $\deg(p_\mathcal{F}) = \operatorname{mult}(p_\mathcal{G}, c)$*;*

(ii) $d_\mathcal{F} = d_\mathcal{G} - (\pi(\mathcal{F}) - \pi_\mathcal{G}) \deg(p_\mathcal{F})$.

Statement 3.18. *Let $\mathcal{F} \in T_a$, $\kappa \in \mathbb{N}^*$ be suitable such that $\kappa\pi(\mathcal{F}) \in \mathbb{N}$. Then, if $\mathcal{F} * c$ exists, then c is a root of $p_\mathcal{F}$.*

If 0 is a root of $p_\mathcal{F}$, then $\mathcal{F} * 0$ exists. If $c \in \mathbb{C}^*$ is a root of $p_\mathcal{F}$, then there exists a unique $\nu_\mathcal{F}$-th root of unity, ε such that $\mathcal{F} * c$ exists.

4 Abhyankar's idea

Proposition 4.1. *Set $\mathcal{F} \in T_{\kappa,a}$ with $\pi(\mathcal{F}) := u$ and assume $b \in \mathbb{C}$ satisfies*

$$(g-b)^+_\mathcal{F} \not\equiv c \in \mathbb{C}. \tag{7}$$

Then $d_\mathcal{F} + d_{g-b,\mathcal{F}} \geq 1 - u$ and

$$J(f^+_\mathcal{F}, (g-b)^+_\mathcal{F}) = \begin{cases} \ominus\xi^{-u} & \text{if } d_\mathcal{F} + d_{g-b,\mathcal{F}} = 1-u \\ 0 & \text{if } d_\mathcal{F} + d_{g-b,\mathcal{F}} > 1-u. \end{cases} \tag{8}$$

Proof. Chose $\kappa \in \mathbb{N}^*$ such that $\kappa u := n \in \mathbb{N}$.

From the chain rule by induction we have $J(f^\mathcal{F}(\xi,\eta), (g-b)(\xi,\eta)^\mathcal{F}) = \ominus\xi^{-n/\kappa}$ (cf. Notation 3.9).

On the other hand, denoting $d_\mathcal{F} + d_{g-b,\mathcal{F}} - 1$ by $\frac{l}{\kappa}$ for some $l \in \mathbb{Z}$, there exists a sequence of polynomials $p_j, p_{j+1}, \ldots, p_l$ with

$$J(f^\mathcal{F}, (g-b)^\mathcal{F}) = \xi^{j/\kappa} p_j(\eta_\mathcal{F}) + \xi^{(j+1)/\kappa} p_{j+1}(\eta_\mathcal{F}) + \cdots + \xi^{l/\kappa} p_k(\eta_\mathcal{F}).$$

This shows both $d_\mathcal{F} + d_{g-b,\mathcal{F}} \geq 1 - u$ and (8). □

By using an idea first observed by Abhyankar [A], we obtain a more general statement.

Proposition 4.2. *Set $\mathcal{F} \in T_a^+$. Set $h_0 = g$.*

Then there exist $m := m_{\mathcal{F}} \in \mathbb{N}$,
$K_{\mathcal{F}} := (k_{0,\mathcal{F}}, \dots, k_{m-1,\mathcal{F}}) := (k_0, \dots, k_{m-1}) \in \mathbb{N}^{*m-1}$,
$L_{\mathcal{F}} := (l_{0,\mathcal{F}}, \dots, l_{m-1,\mathcal{F}}) := (l_0, \dots, l_{m-1}) \in \mathbb{N}^{*\,m-1}$,
$S_{\mathcal{F}} := (s_{0,\mathcal{F}}, \dots, s_{m-1,\mathcal{F}}) := (s_0, \dots, s_{m-1}) \in \mathbb{C}^{*\,m-1}$,
$h_{1,\mathcal{F}} := h_1, \dots, h_{m,\mathcal{F},b} := h_m := h_{\mathcal{F}} : \mathbb{C}^2 \to \mathbb{C}$ *non-constant polynomials with the following properties uniquely defining them:*

(i) $\gcd(k_j, l_j) = 1$*;*

(ii) $h_{j+1} = h_j^{k_j} - s_j f^{l_j}$ *($j = 0, \dots, m-1$);*

(iii) $(h_{j,\mathcal{F}}^+)^{k_j} = s_j (f_{\mathcal{F}}^+)^{l_j}$ *($j = 0, \dots, m-1$);*

(iv) $J(f_{\mathcal{F}}^+, h_{m,\mathcal{F}}^+) = \ominus (f_{\mathcal{F}}^+)^{\mu} \xi^{-u}$*, where $\mu := \mu_{\mathcal{F}} = 0$ if $m = 0$ and $\mu := \mu_{\mathcal{F}} = \frac{(k_0-1)l_0}{k_0} + \dots + \frac{(k_{m-1}-1)l_{m-1}}{k_{m-1}}$ if $m > 0$.*

Proof. By recursion we define the sequences $h_0, \dots, h_j$, $l_0, \dots, l_{j-1}$, $k_0, \dots, k_{j-1}$ and $s_0, \dots, s_{j-1}$ with the properties (i), (ii) and (iii). We have

$$J(f, h_j) = \ominus J(f, h_j) h_{j-1}^{k_{j-1}-1} = \dots = \ominus J(f, h_0) h_0^{k_0-1} \dots h_{j-1}^{k_{j-1}-1} = \ominus h_0^{k_0-1} \dots h_{j-1}^{k_{j-1}-1}.$$

By using the chain rule and the above formula we get

$$J(f^{\mathcal{F}}, h_j^{\mathcal{F}}) = \ominus (h_0^{\mathcal{F}})^{k_0-1} \dots (h_j^{\mathcal{F}})^{k_{j-1}-1} \xi^{-u}.$$

Define

$$\alpha_j = \alpha_{j,\mathcal{F}} := \frac{(k_0-1)l_0}{k_0} + \dots + \frac{(k_{j-1}-1)l_{j-1}}{k_{j-1}}. \tag{9}$$

(By definition, $\alpha_0 = 0$.) By using the argument of the proof of Proposition 4.1 and the two above formulae we obtain

$$d_{\mathcal{F}} + d_{h_j,\mathcal{F}} \geq \alpha_j d_{\mathcal{F}} + 1 - u.$$

Moreover,

$$J(f_{\mathcal{F}}^+, h_{j,\mathcal{F}}^+) = \begin{cases} \ominus (f_{\mathcal{F}}^+)^{\alpha_j} \xi^{-u} & \text{if } \ d_{\mathcal{F}} + d_{h_j,\mathcal{F}} = \alpha_j d_{\mathcal{F}} + 1 - u \\ 0 & \text{if } \ d_{\mathcal{F}} + d_{h_j,\mathcal{F}} > \alpha_j d_{\mathcal{F}} + 1 - u. \end{cases} \tag{10}$$

We consider the two cases of (10). Assume first

$$J(f_{\mathcal{F}}^+, h_{j,\mathcal{F}}^+) = 0.$$

Then it can be easily proved, that there exist k_j, $l_j \in \mathbb{N}^*$ and $s_j \in \mathbb{C}$ with (i) and (iii). Moreover, since h_j is a non-constant polynomial, $s_j \neq 0$ and $l_j \neq 0$ and k_j, l_j are uniquely defined. As we have seen,

$$J(f, h_{j+1}) = \ominus h_0^{k_0-1} \dots h_j^{k_j-1},$$

showing that h_{j+1} cannot be a constant polynomial.

Assume now

$$J(f^+_{\mathcal{F}}, h^+_{j,\mathcal{F}}) = \ominus (f^+_{\mathcal{F}})^{\alpha_j} \xi^{-u}.$$

Then set $m := j$, hence evidently (iv) holds.

To finish the proof we need to prove that our process will stop after finitely many steps. Choose $\kappa \in \mathbb{N}^*$ such that $u\kappa \in \mathbb{N}$. Define

$$\delta_j := \kappa(d_{\mathcal{F}} + d_{h_j,\mathcal{F}} - \alpha_j d_{\mathcal{F}} - 1 + u) \in \mathbb{N}.$$

We have to prove that $\delta_j = 0$ for some j. This follows from the monotone strictly decreasing property of δ_j, which will be proved next.

Let us prove this last property. By (iii) and (i) we have
$d_{h_{j+1},\mathcal{F}} < k_j d_{h_j,\mathcal{F}} = l_j d_{\mathcal{F}}$.

So,

$$\delta_{j+1} = \kappa(d_{\mathcal{F}} + d_{h_{j+1},\mathcal{F}} - \alpha_{j+1} d_{\mathcal{F}} - 1 + u) =$$
$$\kappa(d_{\mathcal{F}} + d_{h_{j+1},\mathcal{F}} - (\alpha_j + \tfrac{(k_j-1)l_j}{k_j}) d_{\mathcal{F}} - 1 + u) = \delta_j + \kappa(d_{h_{j+1},\mathcal{F}} - \tfrac{(k_j-1)l_j d_{\mathcal{F}}}{k_j} - d_{h_j,\mathcal{F}}) =$$
$$\delta_j + \kappa(d_{h_{j+1},\mathcal{F}} - \tfrac{k_j l_j d_{\mathcal{F}}}{k_j}) < \delta_j. \qquad \square$$

Remark: From the above proposition we also obtain that the condition (7) automatically holds in the case $d_{\mathcal{F}} > 0$.

Proposition 4.3. *Set $\mathcal{F} \in T_a^-$. Set $h_0 = g - b$, where $b \in \mathbb{C}$ satisfies (7).*

Then there exist $m := m_{\mathcal{F},b} \in \mathbb{N}$,
$K_{\mathcal{F},b} := (k_{0,\mathcal{F}}, \dots, k_{m-1,\mathcal{F}}) := (k_0, \dots, k_{m-1}) \in \mathbb{N}^{*m-1}$,
$L_{\mathcal{F},b} := (l_{0,\mathcal{F}}, \dots, l_{m-1,\mathcal{F}}) := (l_0, \dots, l_{m-1}) \in \mathbb{N}^{*\,m-1}$,
$S_{\mathcal{F},b} := (s_{0,\mathcal{F}}, \dots, s_{m-1,\mathcal{F}}) := (s_0, \dots, s_{m-1}) \in \mathbb{C}^{*\,m-1}$,
$h_{1,\mathcal{F},b} := h_1, \dots, h_{m,\mathcal{F},b} := h_m := h_{\mathcal{F},b} : \mathbb{C}^2 \to \mathbb{C}$ *non-constant polynomials with the following properties uniquely defining them:*

(i) $\gcd(k_j, l_j) = 1$*;*

(ii) $h_{j+1} = h_j^{k_j} - s_j(f-a)^{l_j}$ *$(j = 0, \dots, m-1)$;*

(iii) $(h^+_{j,\mathcal{F}})^{k_j} = s_j((f-a)^+_{\mathcal{F}})^{l_j}$ *($j = 0, \dots, m-1$);*

(iv) $J((f-a)^+_{\mathcal{F}}, h^+_{m,\mathcal{F}}) = \ominus (f^+_{\mathcal{F}})^{\mu}\xi^{-u}$*, where* $\mu := \mu_{\mathcal{F}} = 0$ *if* $m = 0$ *and* $\mu := \mu_{\mathcal{F}} = \frac{(k_0-1)l_0}{k_0} + \cdots + \frac{(k_{m-1}-1)l_{m-1}}{k_{m-1}}$ *if* $m > 0$*.*

It will turn out, that there exists a correspondence between the structure of $\mathcal{F}$ and the corresponding $m_{\mathcal{F},b}$, $K_{\mathcal{F},b}$ $L_{\mathcal{F},b}$ $S_{\mathcal{F},b}$ and $h_{\mathcal{F},b}$. To see this we introduce a useful notation.

Notation 4.1. *Set* $\mathcal{F}, \mathcal{G} \in T^+_a$*. We introduce* $\mathcal{F} \preceq (\mathcal{G}$ *if and only if*

(i) $m := m_{\mathcal{F}} \leq m_{\mathcal{G}} := m^*$*.*

(ii) If $K_{\mathcal{G}} = (k_0, \dots, k_{m^*-1})$*,* $L_{\mathcal{G}} = (l_0, \dots, l_{m^*-1})$ *and* $S_{\mathcal{G}} = (s_0, \dots, s_{m^*-1})$*, then* $K_{\mathcal{F}} = (k_0, \dots, k_{m-1})$*,* $L_{\mathcal{F}} = (l_0, \dots, l_{m-1})$ *and* $S_{\mathcal{F}} = (s_0, \dots, s_{m-1})$*.*

Furthermore, $\mathcal{F} \prec \mathcal{G}$ *if and only if* $\mathcal{F} \preceq \mathcal{G}$ *and* $m_{\mathcal{F}} < m_{\mathcal{G}}$*.*

Notation 4.2. *Set* $\mathcal{F}, \mathcal{G} \in T^-_a$*,* $b \in \mathbb{C}$*, and assume* $b \in \mathbb{C}$ *satisfies (7). We introduce* $(\mathcal{F}, b) \preceq (\mathcal{G}, b)$ *if and only if*

(i) $m := m_{\mathcal{F},b} \leq m_{\mathcal{G},b} := m^*$*.*

(ii) If $K_{\mathcal{G},b} = (k_0, \dots, k_{m^*-1})$*,* $L_{\mathcal{G},b} = (l_0, \dots, l_{m^*-1})$ *and* $S_{\mathcal{G},b} = (s_0, \dots, s_{m^*-1})$*, then* $K_{\mathcal{F},b} = (k_0, \dots, k_{m-1})$*,* $L_{\mathcal{F},b} = (l_0, \dots, l_{m-1})$ *and* $S_{\mathcal{F},b} = (s_0, \dots, s_{m-1})$*.*

Furthermore, $(\mathcal{F}, b) \prec (\mathcal{G}, b)$ *if and only if* $(\mathcal{F}, b) \preceq (\mathcal{G}, b)$ *and* $m_{\mathcal{F}} < m_{\mathcal{G}}$*.*

Proposition 4.4. *Set* $\mathcal{F} \in T_a$ *and take* $\kappa \in \mathbb{N}^*$ *be suitable such that* $\kappa\pi(\mathcal{F}) \in \mathbb{N}^*$*. Let* $h(x, y)$ *be a polynomial. Assume that there exist* $k, l, k^*, l^* \in \mathbb{N}^*$ *with* $\gcd(k, l) = \gcd(k^*, l^*) = 1$ *and* $s, s^* \in \mathbb{C} \setminus \{0\}$ *with* $(h^+_{\mathcal{F}})^k = s(f^+_{\mathcal{F}})^l$ *and* $(h^+_{\mathcal{F}'})^{k^*} = s^*(f^+_{\mathcal{F}'})^{l^*}$*, and assume that one of the following properties holds:*

(i) $\mathcal{F} \in T^+_a$

(ii) $\mathcal{F}' \in T^-_a$*.*

Then $k = k^*$*,* $l = l^*$ *and* $s = s^*$*.*

In particular, for any $\mathcal{F} \in T^+_a$*, either* $\mathcal{F} \preceq \mathcal{F}'$ *or* $\mathcal{F}' \preceq \mathcal{F}$*, and for any* $\mathcal{F}, \mathcal{F}' \in T^-_a$ *and for any* $b \in \mathbb{C}$ *satisfying (7), either* $(\mathcal{F}, b) \preceq (\mathcal{F}', b)$ *or* $(\mathcal{F}', b) \preceq (\mathcal{F}, b)$*.*

Proof. Set $\mathcal{F} = \mathcal{F}' * c$. By Statement 3.18, c is a root of $p := p_{\mathcal{F}'}$. Write $p(\eta) = \sum_{j=\alpha}^{m} a_j(\eta - c)^j$, with $a_\alpha \neq 0$, and $\alpha > 0$.

Set $q = p_{h,\mathcal{F}'}$. By the assumptions of our statement,

$$q^{k^*}(\eta) = s^* p^{l^*}(\eta) = s^* \left(\sum_{j=\alpha}^{m} a_j (\eta - c)^j \right)^{l^*},$$

hence $q(\eta) = \sum_{j=\beta}^{M} A_j (\eta - c)^j$, where $\beta = \frac{l^*}{k^*}\alpha$ and $A_\beta = \sqrt[k^*]{s^* a_\alpha^{l^*}}$ for a suitable choice of $\sqrt[k^*]{s^* a_\alpha^{l^*}}$.

We state that $\deg(p_{\mathcal{F},h}) = \mathrm{mult}(q, c_n)$. Assume first $\mathcal{F} \in T_a^+$. From Statement 3.9 we have

$$p_{\mathcal{F}}(\eta) = \sum_{j=0}^{\alpha} b_j \eta^j,$$

where $b_\alpha = a_\alpha$.

From our assumption, and by Statements 3.9 and 3.11,

$$\frac{\deg(p_{\mathcal{F}})}{\kappa d_{\mathcal{F}}} = \frac{\deg(p_{h,\mathcal{F}})}{\kappa d_{h,\mathcal{F}}} \leq \frac{\mathrm{mult}(q, c)}{\kappa d_{h,\mathcal{F}'} - \mathrm{mult}(q, c_n)} = \frac{\mathrm{mult}(p, c)}{\kappa d_{\mathcal{F}'} - \mathrm{mult}(p, c_n)} = \frac{\deg(p_{\mathcal{F}})}{\kappa d_{\mathcal{F}}}.$$

Therefore, by Statement 3.11, $\deg(p_{h,\mathcal{F}}) = \mathrm{mult}(q, c)$.

Assume now $\mathcal{F}' \in T_a^-$, and assume indirectly $\deg(p_{h,\mathcal{F}}) \neq \mathrm{mult}(q, c_n)$. Then

$$\frac{\kappa d_{h,\mathcal{F}}}{\deg(p_{h,\mathcal{F}})} = \frac{\kappa d_{\mathcal{F}}}{\deg(p_{\mathcal{F}})} = \frac{\kappa d_{\mathcal{F}'} - \mathrm{mult}(p, c_n)}{\mathrm{mult}(p, c_n)} = \frac{\kappa d_{h\mathcal{F}'} - \mathrm{mult}(q, c_n)}{\mathrm{mult}(q, c_n)}.$$

From Statement 3.11, there exist $\lambda \in [0, 1]$ with

$$\kappa d_{h,\mathcal{F}'} = \kappa d_{h,\mathcal{F}} + \lambda \mathrm{mult}(q, c_n) + (1 - \lambda) \deg(p_{h,\mathcal{F}}).$$

By using the two above formulae, we obtain

$$(\mathrm{mult}(q, c_n) - \deg(p_{h,\mathcal{F}}))\kappa d_{h,\mathcal{F}} = (\lambda - 1) \deg(p_{h,\mathcal{F}})(\mathrm{mult}(q, c_n) - \deg(p_{h,\mathcal{F}})).$$

Our indirect assumption says $\mathrm{mult}(q, c_n) - \deg(p_{h,\mathcal{F}}) \neq 0$, so we obtain $\kappa d_{h,\mathcal{F}} = (\lambda - 1) \deg(p_{h,\mathcal{F}})$, hence $\kappa d_{\mathcal{F}} = (\lambda - 1) \deg(p_{\mathcal{F}})$, which gives $\kappa d_{\mathcal{F}} \geq -\deg(p_{\mathcal{F}})$. Therefore, by Statement 3.9, $d_{\mathcal{F}'} \geq 0$, which is a contradiction.

From Statement 3.11 both in the case $\mathcal{F} \in T_a^+$ and $\mathcal{F}' \in T_a^-$, $p_{h,\mathcal{F}}(\eta) = \sum_{j=0}^{\beta} B_j (\eta)^j$, where $B_\beta = A_\beta$. Since $p_{h,\mathcal{F}}^k = s p_{\mathcal{F}}^l$, we have $k\beta = l\alpha$. By using $\gcd(k, l) = \gcd(k^*, l^*) = 1$, we have $k = k^*$, $l = l^*$. Moreover, $B_\beta^k = s b_\alpha^l$, showing $s = s^*$. $\square$

Proposition 4.5. *Let h be a polynomial. Assume that there exist $k, l, k^*, l^* \in \mathbb{N}^*$ with $\gcd(k, l) = \gcd(k^*, l^*) = 1$ and $s, s^* \in \mathbb{C}^*$ with $h_{(0,x)}^k = s(f_{(0,x)}^+)^l$ and $h_{(0,y)}^{k^*} = s^*(f_{(0,y)}^+)^{l^*}$. Then $k = k^*$, $l = l^*$ and $s = s^*$.*

In particular, either $(0, x) \preceq (0, y)$ or $(0, y) \preceq (0, x)$.

Proof. By our assumption we have $\frac{l}{k}(k_f, l_f) \in N_h$ and $\frac{l^*}{k^*}(k_f, l_f) \in N_h$. Moreover, if $(i, j) \in N_h$, then $i \le \frac{l}{k} k_f$ and $j \le \frac{l^*}{k^*} l_f$ showing $\frac{l}{k} = \frac{l^*}{k^*}$. The remaining part follows from Statement 3.12. $\square$

Proposition 4.6. *Set $\mathcal{F} = T_a^+ \cup T_a^-$. Assume $b \in \mathbb{C}$ satisfies (7). Set $h = h_{\mathcal{F},b}$. Set $p = p_{\mathcal{F}}$, $q = p_{h,\mathcal{F}}$, $\mu = \mu_{\mathcal{F}}$ $k = d_{\mathcal{F}}$, $l = d_{h,\mathcal{F},b}$. Then (10) gives*

$$kpq' - lp'q = \ominus p^{\mu}. \tag{11}$$

In particular, (11) has the following consequences regarding the multiplicities of the roots of p and q. Let c^* be a root of $p_{\mathcal{F}}$. Then either

$$\mathrm{mult}(p, c^*) + \mathrm{mult}(q, c^*) - 1 < \mu_{\mathcal{F}} \mathrm{mult}(p, c^*), \tag{12}$$

and in this case

$$\frac{\mathrm{mult}(p, c^*)}{\mathrm{mult}(q, c^*)} = \frac{d_{\mathcal{F}}}{d_{h,\mathcal{F}}}, \tag{13}$$

or

$$\mathrm{mult}(p, c^*) + \mathrm{mult}(q, c^*) - 1 = \mu_{\mathcal{F}} \mathrm{mult}(p, c^*). \tag{14}$$

Moreover, if (14) holds, then (13) is not true.

Similarly the degrees of the polynomials p and q satisfy either

$$\deg p + \deg q - 1 > \mu_{\mathcal{F}} \deg p, \tag{15}$$

and in this case

$$\frac{\deg p}{\deg q} = \frac{d_{\mathcal{F}}}{d_{h,\mathcal{F}}}, \tag{16}$$

or

$$\deg p + \deg q - 1 = \mu_{\mathcal{F}} \deg p. \tag{17}$$

Moreover, if (17) holds, then (16) is not true.

5 The poles of g

The non-constant meromorphic function $g : \overline{R}_a \to \mathbb{C} \cup \{\infty\}$ takes at some points the value ∞, or, in another language it has poles. As we have established, these points are elements of $\overline{R}_a \setminus R_a$. Hence, it is natural to ask, how can we describe the set of g-poles from the properties of T_a.

Proposition 5.1. *Set $P \in \overline{R}_a \setminus R_a$. Then there exists $u \in \mathbb{Q}^+$ with the following properties:*

(i) For $\mathcal{F} = I_P(u)$, $m_{\mathcal{F}} = 0$.

(ii) For any $u^* \in \mathbb{Q}$ with $u^* < u$ and for $\mathcal{G} := I_P(u^*)$, $m_{\mathcal{G}} \neq 0$.

(iii) For any u^* with $u^* \geq u$ and for $\mathcal{G} = I_P(u^*)$, $m_{\mathcal{G}} = 0$.

Proof. Define $\mathcal{F}_v := I_P(v)$ for any $v \in \mathbb{Q}$. From Statements 3.10, 3.11 and 3.18, we obtain that $\rho(v) := d_{\mathcal{F}_v} + d_{g,\mathcal{F}_v} + v - 1$ is monotone decreasing and continuous, and except of finitely many rational points, ρ is locally linear. Moreover, from Proposition 4.1, we have $\rho(v) \geq 0$. This shows, that if ρ has a zero, then it has a minimal rational zero point, and if $\rho(v) = 0$, then, for any $w \in \mathbb{Q}^+$, $\rho(v + w) = 0$. By Proposition 8, $\rho(v) = 0$ if and only if $m_{\mathcal{F}_v} = 0$.

Our only task is to prove that ρ takes the value 0. Assume indirectly that for any $v \in \mathbb{Q}^+$, $\rho(v) \neq 0$. Then from Proposition 8 we obtain $m_{\mathcal{F}_v} > 0$. From Proposition 4.2, $(g^+_{\mathcal{F}_n})^l = \ominus(f^+_{\mathcal{F}_n})^k$, so by Statement 3.18, $\deg(p_{\mathcal{F}_v}), \deg(p_{g,\mathcal{F}_v}) \geq 1$. This however gives that for any $w \in \mathbb{Q}^+$, $\rho(v + w) \leq \rho(v) - w$, so if w is large enough, then $\rho(v + w) < 0$, which is a contradiction. ☐

Proposition 5.2. *There exists $\mathcal{F} \in T^+_a$ with $m_{\mathcal{F}} = 0$.*

Proof. Assume indirectly that $m_{\mathcal{F}} > 0$ for each $\mathcal{F} \in T^+_a$. Then by Propositions 4.2, 4.4 and 4.5, there exist $k, l \in \mathbb{N}^*$ with $(g^+_{\mathcal{F}})^k = \ominus(f^+_{\mathcal{F}})^l$ for each $\mathcal{F} \in T^+_a$. Consequently, $kd_{g,\mathcal{F}} = ld_{\mathcal{F}}$. By Statement 3.9 this gives $d_{\mathcal{F}} > 0$ if and only if $d_{g,\mathcal{F}} > 0$. By Statement 3.15 we obtain that g has no pole on $\overline{R}_a$, so g is a constant meromorphic function on $\overline{R}_a$, which contradicts the inverse function's theorem. ☐

Notation 5.1. *Set $P \in \overline{R}_a \setminus R_a$. Assume $u \in \mathbb{Q}^+$ satifies (i), (ii) and (iii) of Proposition 5.1. Define $\mathcal{F}^*_P := I_P(u)$.*

Statement 5.1. *Set $P \in \overline{R}_a \setminus R_a$, $\mathcal{F} := \mathcal{F}^*_P$. Then $d_{g\mathcal{F}} > 0$ if and only if $d_{\mathcal{F}} > 0$.*

Proof. Set $\mathcal{F} := I_P(u)$. Assume first $d_{\mathcal{F}} > 0$. Then for any $v \in \mathbb{Q}$ with $v < u$ for $\mathcal{G} := I_P(v)$ we have $d_{\mathcal{G}} > 0$, $m_{\mathcal{G}} > 0$, so, from Proposition 4.2, there exist $k, l \in \mathbb{N}$ with $\gcd .(k, l) = 1$ and with $(g^+_{\mathcal{G}})^k = \ominus(f^+_{\mathcal{G}})^l$. By Proposition 4.4 we have that k, l don't depend on the choice of v. Therefore $\frac{d_{g,\mathcal{G}}}{d_{\mathcal{G}}} = \frac{k}{l}$ is a constant function of v. From (ii) of Proposition 3.10, we obtain $\frac{d_{g,\mathcal{F}}}{d_{\mathcal{F}}} = \frac{k}{l}$, showing $d_{\mathcal{F}} > 0$.

Assume $d_{\mathcal{F}} \leq 0$. Set $v \in \mathbb{Q}^+$, $v \leq u$ such that for $\mathcal{G} = I_P(v)$, $d_{\mathcal{G}} = 0$. Then, for any $w \in \mathbb{Q}$ with $w < v$ for $\mathcal{H} := I_P(w)$, we have $d_{\mathcal{H}} > 0$ and $m_{\mathcal{H}} > 0$. In the same way as above, we obtain that $\frac{d_{g,\mathcal{H}}}{d_{\mathcal{H}}} = \frac{k}{l}$ is a constant function, so as above, $d_{g,\mathcal{G}} = 0$. By (i) of Statement 3.10, $d_{g,\mathcal{F}} \leq 0$. ☐

Notation 5.2. *Define*

$$T_{a,\mathrm{pole}} := \{\mathcal{F}_P^* : \ P \in \overline{R}_a \setminus R_a\} \cap T_a^+.$$

Proposition 5.3. *Set* $\mathcal{F} \in T_{a,\mathrm{pole}}$*,* $\mathcal{F} := I_P(u)$*. Then*

(i) $\mathcal{F} \neq (0, x)$ *and* $\mathcal{F} \neq (0, y)$*.*

(ii) $d_{g,\mathcal{F}} > 0$*. Moreover,* $d_{g,\mathcal{F}} = \frac{k_f}{k_g} d_{\mathcal{F}}$*.*

(iii) $kp_{\mathcal{F}}p'_{g,\mathcal{F}} - lp'_{\mathcal{F}}p_{g,\mathcal{F}} = \ominus$*, where* $k = k_f$*,* $l = k_g$*.*

(iv) $\mathcal{F} \in V_a$*.*

(v) For any $c \in \mathbb{C}$*,* $\mathrm{mult}(p_{\mathcal{F}}, c) \leq 1$ *and* $\mathrm{mult}(p_{g,\mathcal{F}}, c) \leq 1$*.*

(vi) For any $c \in \mathbb{C}$*,* $\mathrm{mult}(p_{g,\mathcal{F}}) \cdot \mathrm{mult}(p_{\mathcal{F}}, c) = 0$*.*

(vii) $\frac{\deg(p_{\mathcal{F}})}{\deg(p_{g,\mathcal{F}})} = \frac{d_{\mathcal{F}}}{d_{g,\mathcal{F}}}$*.*

(viii) For any $v \in \mathbb{Q}^+$*,* $v \leq u$ *and for* $\mathcal{G} = I_P(v)$*,* $d_{g,\mathcal{G}} = \frac{k}{l} d_{\mathcal{G}}$*.*

(ix) $I_P((u, \infty]) \cap V_a = \emptyset$*.*

Proof. If $\mathcal{F} = (0, x)$ or $\mathcal{F} = (0, y)$, then by the definition (and from the fact that both x and y degrees of f and g are positive), by (8) we have $J(f_{\mathcal{F}}^+, g_{\mathcal{F}}^+) = 0$, showing (i).

Now we prove (ii). Set $\mathcal{F} = I_P(u)$. From the definition, for any $v \in \mathbb{Q}^+$ with $v < u$ and $\mathcal{G} = I_P(v)$ we have $(g_{\mathcal{G}}^+)^k = \ominus(f_{\mathcal{G}}^+)^l$ with $k, l \in \mathbb{N}^*$ and $\gcd.(k, l) = 1$. From Proposition 4.4 we have that k and l are independent of the choice of v, hence $\frac{k}{l} = \frac{k_f}{k_g}$. Therefore $d_{g,\mathcal{G}} = \frac{k_f}{l_f} d_{\mathcal{G}}$. From (ii) of Statement 3.10 this gives (ii).

From Proposition 4.6 and from (iv) from Proposition 4.2, we have $kpq' - lp'q = \ominus$, where $p = p_{\mathcal{F}}$ and $q = p_{g,\mathcal{F}}$, which is exactly (iii).

From (iii), for any $c \in \mathbb{C}$ one has $\mathrm{mult}(p, c) \leq 1$ (which is (v)), and $\mathrm{mult}(p, c)\mathrm{mult}(q, c) = 0$. By Statement 3.11, this gives that for any $\nu \in \mathbb{N}^*$ and for any $\mathcal{G} \in T_a$ with $\mathcal{G}^{(\nu)} = \mathcal{F}$, $\deg(p_{\mathcal{G}}) \leq 1$, $\deg(p_{g,\mathcal{G}}) = 0$ and $d_{g,\mathcal{G}} \geq d_{g,\mathcal{F}}$. By Statement 3.9, $\deg(p_{\mathcal{G}}) > 0$, so $\deg(p_{\mathcal{G}}) = 1$. Moreover, by the definition, $d_{g,\mathcal{G}} \leq d_{g,\mathcal{F}}$, so $d_{g,\mathcal{G}} = d_{g,\mathcal{F}}$.

Let us prove (iv). By Statement 3.16, we have to prove, that $p_{\mathcal{F}}$ has more than one root. Otherwise, by (iii), $p_{\mathcal{F}}$ would be linear, and $\frac{k_g}{k_f} \in \mathbb{N}$, contradicting Statement 2.1. $\square$

Proposition 5.4. *Set $\mathcal{F} \in T_{a,\mathrm{pole}}$. Set $p := p_{\mathcal{F}}$, $q := p_{g,\mathcal{F}}$. Assume $\nu_{\mathcal{F}} := \nu \neq 1$. Then one of the following statements hold*

(i) $p(\eta) = p^*(\eta^\nu)$ and $q(\eta) = \eta q^*(\eta^\nu)$ for some polynomials p^* and q^*.

(ii) $p(\eta) = \eta p^*(\eta^\nu)$ and $q(\eta) = q^*(\eta^\nu)$ for some polynomials p^* and q^*.

Proof. From (iv) of Proposition 5.3 and from Statement 3.16, we have $p(\eta) = \eta^k p^*(\eta^\nu)$. By (v) of Proposition 5.3, either $k = 0$, or $k = 1$. By (ii), the $k = 0$ case gives (i), the $k = 1$ case gives (ii). $\square$

Notation 5.3. *Set $\mathcal{F} \in T_{a,\mathrm{pole}}$. The following data is called the multiplicity data:*

(i) The type of (f, g), (α, β);

(ii) $(D_{g,\mathcal{F}}, D_{\mathcal{F}})$;

(iii) $\nu_{\mathcal{F}}$;

(iv) $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}}))$.

Statement 5.2. *The multiplicity data satisfies the following properties:*

(i) $\frac{\deg(p_{\mathcal{F}})}{\deg(p_{g,\mathcal{F}})} = \frac{D_{\mathcal{F}}}{D_{g,\mathcal{F}}} = \frac{\alpha}{\beta}$.

(ii) Either $\nu_{\mathcal{F}} \mid \alpha$, $\nu_{\mathcal{F}} \mid \deg(p_{g,\mathcal{F}}) - 1$ or $\nu_{\mathcal{F}} \mid \beta$, $\nu_{\mathcal{F}} \mid \deg(p_{\mathcal{F}}) - 1$.

Proof. (i) follows from (ii) and (vii) of Proposition 5.3.

By Proposition 5.4, either $\nu_{\mathcal{F}} \mid \deg(p_{\mathcal{F}})$ and $\nu_{\mathcal{F}} \mid \deg(p_{g,\mathcal{F}}) - 1$, or $\nu_{\mathcal{F}} \mid \deg p_{g,\mathcal{F}}$ and $\nu_{\mathcal{F}} \mid \deg(p_{\mathcal{F}}) - 1$. In the first case, by Proposition 5.3 one has

$$\nu_{\mathcal{F}} \mid \deg(p_{\mathcal{F}}) = \alpha \frac{\deg(p_{g,\mathcal{F}})}{\beta}.$$

Since $gcd.(\deg(p_{g,\mathcal{F}}), \nu_{\mathcal{F}}) = 1$, we have $\nu_{\mathcal{F}} \mid \alpha$. In the second case the same type of calculation works. $\square$

We have the next formula for the order of the poles of g on R_a.

Proposition 5.5. *Set $P \in \overline{R}_a \setminus R_a$. Then $g(P) = \infty$ if and only if there exists $u \in \mathbb{Q}^+$ such that $\mathcal{F} := I_P(u) \in T_{a,\mathrm{pole}}$.*

Let $\kappa \in \mathbb{N}^*$ be suitable, and assume that for $c \in \mathbb{C}$, $\mathcal{F} * c = I_P(u + \frac{1}{\kappa})$. Then the order of the pole at $\mathcal{P}$ is obtained by the following formula

$$\Lambda(P) = \begin{cases} \frac{D_{g,\mathcal{F}}}{\nu_{\mathcal{F}}} & \text{if } c = 0 \\ D_{g,\mathcal{F}} & \text{if } c \neq 0 \end{cases} \tag{18}$$

Proof. From Statements 3.15 and 5.1 we obtain that $g(P) = \infty$ if and only if $\mathcal{F}_P^* \in T_{a,\text{pole}}$, which is equivalent with the existenece of u with $I_P(u) \in T_{a,\text{pole}}$.

Assume that the Puiseux characteristics of P is $(\kappa, \beta_1, \dots, \beta_m)$. Let v be the maximal element of $V_{1,a} \cap I_P([0, u])$. Then there are two cases:

(i) $v < u$.

(ii) $v = u$.

In the case (i), using (ix) of Proposition 5.3, for any $Q \in \overline{R}_a \setminus R_a$, with $I_Q(v) = I_P(v)$, the Puiseux characteristics of Q is $(\kappa, \beta_1, \dots, \beta_m)$, and $\alpha_m = v$. Therefore $\nu_{\mathcal{F}} = 1$ and from Statement 3.15, $D_{g,\mathcal{F}} = \kappa d_{\mathcal{F}} = \Lambda(P)$, so, in this case we obtain (18).

Consider the case (ii). Assume first $c \neq 0$. Then $u = \alpha_m$, hence by Statement 18 one gets $\Lambda(P) = \kappa d_{\mathcal{F}} = D_{\mathcal{F}}$.

Assume $c = 0$. Then $\alpha_m < u$. Moreover, for any $Q \in \overline{\mathbb{R}}_a(\setminus R_a \cup \{P\})$, with $I_Q(u) = \mathcal{F}$, the Puiseux characteristics of Q is $(\nu_{\mathcal{F}}\kappa, \beta_1, \dots, \beta_m, u)$. In this case $\Lambda(P) = \kappa d_{g,\mathcal{F}} = \frac{D_{\mathcal{F}}}{\nu_{\mathcal{F}}}$. □

Proposition 5.6. *Set $\mathcal{F} \in T_{a,\text{pole}}$ and $u = \pi(\mathcal{F})$. Then*

$$\Lambda(\mathcal{F}) := \frac{D_{g,\mathcal{F}} \deg(p_{\mathcal{F}})}{\nu_{\mathcal{F}}} = \sum_{P \in \overline{R}_a \setminus R_a:\ \mathcal{F} = I_P(u)} \Lambda(P). \tag{19}$$

Proof. Let $\kappa \in \mathbb{N}^*$ be suitable. Let c be a root of $p_{\mathcal{F}}$. If $c = 0$, then $\mathcal{F} * c$ is defined. If $c \neq 0$, then for any $\nu_{\mathcal{F}}$-th root of unity, ε, εc is a root of $p_{\mathcal{F}}$, and there is a unique ε such that $\mathcal{F} * \varepsilon c$ is defined. Therefore (18) implies (19). □

Proposition 5.7. *Assume that the type of the normalized counterexample (f, g) is (α, β). Set $\mathcal{F} \in T_{a,\text{pole}}$. Then $\Lambda(\mathcal{F}) \geq \beta$.*

Proof. From (ii) of Statement 5.2 we obtain, that at least one of these two possibilities must hold:

(i) $\nu_{\mathcal{F}} \mid \alpha$

(ii)$\nu_{\mathcal{F}} \mid \beta$.

If (i) holds, then

$$\Lambda(\mathcal{F}) = \frac{D_{g,\mathcal{F}} \deg(p_{\mathcal{F}})}{\nu_{\mathcal{F}}} \geq \frac{D_{g,\mathcal{F}} \deg(p_{\mathcal{F}})}{\alpha} \geq \beta.$$

If (ii) holds, then $\nu_{\mathcal{F}} < \deg(p_{\mathcal{F}})$, so

$$\Lambda(\mathcal{F}) = \frac{D_{g,\mathcal{F}} \deg(p_{\mathcal{F}})}{\nu_{\mathcal{F}}} > D_{g,\mathcal{F}} \geq \beta. \qquad \square$$

Definition 5.1. *Let $f^*, g^* : \mathbb{C}^2 \to \mathbb{C}$ be polynomials. The number of the preimages $(f^*, g^*)^{-1}((x, y))$ for generic $(x, y) \in \mathbb{C}^2$ is called the topological degree of (f^*, g^*). It is denoted by $td(f^*, g^*)$.*

Proposition 5.8. *For any $a \in \mathbb{C}$ we have*

$$\operatorname{td}(f, g) = \sum_{\mathcal{F} \in T_{a,\text{pole}}} \frac{D_{g,\mathcal{F}} \deg(p_{\mathcal{F}})}{\nu_{\mathcal{F}}}. \tag{20}$$

6 The tree structure

The ordering of $[0, \infty]$ defines a natural ordering on the Eggers – Wall tree. Recall that in Notation 4.1 we introduced another ordering, which will be rather usefull. In this section we investigate the properties of this ordering.

Proposition 6.1. *Set $\mathcal{F} \in T_a^+ \cup T_a^-$, $u := \pi(\mathcal{F})$. Then $d_{\mathcal{F}} \neq (1-u) \deg(p_{\mathcal{F}})$.*

Proof. Assume indirectly that $d_{\mathcal{F}} = (1 - u) \deg(p_{\mathcal{F}})$. We use the notations of Proposition 4.2 and Notation **??** with the simplification $h := h_{\mathcal{F}}$, $p := p_{\mathcal{F}}$, $q := p_{h,\mathcal{F}}$. From (iv) of Proposition 4.2 we have $d_{h,\mathcal{F}} = 1-u+(\mu_{\mathcal{F}}-1)d_{\mathcal{F}}$. Therefore,

$$\frac{\deg(q)}{\deg(p)} = \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \Leftrightarrow \frac{\deg(q)}{\deg(p)} = \frac{(\mu_{\mathcal{F}} - 1 + \frac{1}{\deg(p)})d_{\mathcal{F}}}{d_{\mathcal{F}}} \Leftrightarrow \deg(q) = (\mu_{\mathcal{F}}-1)\deg(p)+1$$

and this contradicts the last statement of Proposition 4.6. $\square$

As a simple corollary we obtain the following result.

Theorem 6.1. *$l_f < k_f$.*

Proof. Property (iii) of Lemma 2.1 says $l_f \leq k_f$. Moreover, $k_f = d_{(0,x)}$ and $l_f = \deg(p_{(0,x)})$, so $k_f \neq l_f$, by Proposition 6.1. $\square$

Set $\mathcal{F} \in T_a^+$. We want to know whether or not exists $P \in \overline{R}_a \setminus R_a$ such that $\mathcal{F} \in I_P(\mathbb{Q}^+)$ and $g(P) = \infty$. It will turn out, that the following notation leads to the answer of this question.

Notation 6.1. *Define*

$$T_a^{\nearrow} := \{\mathcal{F} \in T_a^+ : \ d_{\mathcal{F}} > (1 - \pi(\mathcal{F})) \deg(p_{\mathcal{F}})\}$$

$$T_a^{\searrow} := \{\mathcal{F} \in T_a^+ : \ d_{\mathcal{F}} < (1 - \pi(\mathcal{F})) \deg(p_{\mathcal{F}})\}.$$

Statement 6.1. *For any $\mathcal{F} \in T_a^+$, either $\mathcal{F} \in T_a^{\searrow}$ or $\mathcal{F} \in T_a^{\nearrow}$*

Proof. The Statement follows from Proposition 6.1. □

Statement 6.2. *Set $\mathcal{F}, \mathcal{G} \in V_a \cap T_a^+$. Assume $\mathcal{F} = \mathcal{G} + c$ for some $c \in \mathbb{C}$. Then $\mathcal{F} \in T_a^{\searrow}$ if and only if $d_{\mathcal{G}} < (1 - \pi(\mathcal{G}))\mathrm{mult}(p_{\mathcal{G}}, c)$. In particular, if $\mathcal{H} = \mathcal{G} + c^*$ for some $c^* \in \mathbb{C}$ and $\mathrm{mult}(p_{\mathcal{G}}, c^*) \geq \mathrm{mult}(p_{\mathcal{G}}, c)$, and $\mathcal{F} \in T_a^{\searrow}$, then $\mathcal{H} \in T_a^{\searrow}$.*

Proof. From Statement 3.17 one has

$$\begin{gathered} d_{\mathcal{F}} - (1 - \pi(\mathcal{F})) \deg(p_{\mathcal{F}}) = d_{\mathcal{G}} - (\pi(\mathcal{F}) - \pi(\mathcal{G})) \deg(p_{\mathcal{F}}) - (1 - \pi(\mathcal{F})) \deg(p_{\mathcal{F}}) = \\ d_{\mathcal{G}} - (1 - \pi(\mathcal{G})) \deg(p_{\mathcal{F}}) = d_{\mathcal{G}} - (1 - \pi(\mathcal{G}))\mathrm{mult}(p_{\mathcal{G}}, c) \end{gathered} .$$

Therefore $d_{\mathcal{F}} < (1-\pi(\mathcal{F})) \deg(p_{\mathcal{F}})$ if and only if $d_{\mathcal{G}} < (1-\pi(\mathcal{G}))\mathrm{mult}(p_{\mathcal{G}}, c)$.□

Proposition 6.2. *Let $\kappa \in \mathbb{N}^*$ be suitable, $\mathcal{F} \in T_a$, and $b \in \mathbb{C}$ satisfying (7). Let $h = h_{\mathcal{F},b}$, $p := p_{\mathcal{F}}$, $q := p_{h,\mathcal{F}}$ and c be a root of p such that $\mathcal{G} := \mathcal{F} * c$ exists. Assume $\pi(\mathcal{G}) \neq 1$ and $\deg(p_{\mathcal{G}}) > \frac{d_{\mathcal{G}}}{1-\pi(\mathcal{G})} > 0$, and assume that one of the following conditions hold:*

(i) $\mathcal{G} \in T_a^-$;

(ii) $\mathcal{F} \in T_a^+$.

Then (14) holds, and $(\mathcal{G}, b) \preceq (\mathcal{F}, b)$.

Proof. Set $m := m_{\mathcal{F},b}$ and $m^* = m_{\mathcal{G},b}$. The second part of our statement is equivalent with $m^* \leq m$. Set $j < m^*$, and $j \leq m$. Our task is to prove $j < m$. From Proposition 4.4, $h_{j,\mathcal{F},b} = h_{j,\mathcal{G},b}$ (cf. Proposition 4.2).

From Proposition 4.2

$$\frac{\deg(p_{\mathcal{G}})}{d_{\mathcal{G}}} = \frac{\deg(p_{h_j,\mathcal{G}})}{d_{h_j,\mathcal{G}}}.$$

From the above formula , from Statements 3.9, 3.11 and from (10) we obtain

$$d_{\mathcal{F}} + d_{h_j,\mathcal{F}} \geq d_{\mathcal{G}} + \tfrac{\deg(p_{\mathcal{G}})}{\kappa} + d_{h,\mathcal{G}} + \tfrac{\deg(p_{h,\mathcal{G}})}{\kappa} \geq \tfrac{d_{\mathcal{F}}}{d_{\mathcal{G}}}(d_{\mathcal{G}} + d_{h_j,\mathcal{G}}) >$$
$$\tfrac{d_{\mathcal{F}}}{d_{\mathcal{G}}}(\alpha_j d_{\mathcal{G}} + 1 - \pi(\mathcal{G})) = \alpha_j d_{\mathcal{F}} + \tfrac{d_{\mathcal{F}}}{d_{\mathcal{G}}}(1 - \pi(\mathcal{G})) =$$
$$\alpha_j d_{\mathcal{F}} + 1 - \pi(\mathcal{G}) + \tfrac{\deg(p_{\mathcal{G}})}{\kappa d_{\mathcal{G}}}(1 - \pi(\mathcal{G})) > \alpha_j d_{\mathcal{F}} + 1 - \pi(\mathcal{G}) + \tfrac{1}{\kappa} = \alpha_j d_{\mathcal{F}} + 1 - \pi(\mathcal{F}),$$

so $j < m$ by (10).

The only remaining part is that (14) holds. Suppose indirectly that (13) and (12) hold. Then, from our assumptions,

$$\frac{d_{\mathcal{F}}}{d_{\mathcal{G}}}(1 - \pi(\mathcal{G})) = \frac{d_{\mathcal{G}} + \deg(p_{\mathcal{G}})}{d_{\mathcal{G}}}(1 - \pi(\mathcal{G})) > 1 - \pi(\mathcal{G}) + 1.$$

From (10) we have

$$1 - \pi(\mathcal{F}) = 1 - \pi(\mathcal{G}) + \frac{1}{\kappa} = (1 - \mu)d_{\mathcal{F}} + d_{h_m,\mathcal{F}},$$

hence

$$\frac{d_{\mathcal{G}}}{d_{\mathcal{F}}}(1 - \mu)d_{\mathcal{F}} + d_{h_m,\mathcal{F}} < (1 - \mu)d_{\mathcal{F}} + d_{h_m,\mathcal{F}} - \frac{1}{\kappa}.$$

From (13) we obtain

$$(1 - \mu)d_{\mathcal{G}} + d_{h_m,\mathcal{G}} < (1 - \mu)d_{\mathcal{F}} + d_{h_m,\mathcal{F}} - \frac{1}{\kappa},$$

hence

$$0 < (1 - \mu)\mathrm{mult}(p, c) + \mathrm{mult}(q, c) - 1,$$

contradicting (12). □

Proposition 6.3. *Let $\kappa \in \mathbb{N}^*$ be suitable. Set $\mathcal{F} \in T_a^+$. Set $h = h_{\mathcal{F}}$. Set $p = p_{\mathcal{F}}$, $q = p_{h,\mathcal{F}}$. Assume that $\mathcal{G} := \mathcal{F} * c$ exists for some$c \in \mathbb{C}$.*

Assume $\mathcal{G} \in T_a^{\nearrow}$. Then $\mathcal{F} \preceq \mathcal{G}$. Moreover if (12) also holds, then $\mathcal{F} \prec \mathcal{G}$.

Assume $\mathcal{G} \in T_a^{\searrow}$. Then (14) holds, and $\mathcal{G} \preceq \mathcal{F}$.

Proof. The statements corresponding to $\mathcal{G} \in T_a^{\searrow}$ follows from Proposition 6.2.

Assume $\mathcal{G} \in T_a^{\nearrow}$. We use the notations of Proposition 4.2. Set $j < m_{\mathcal{F}} := m$ and $j \leq m_{\mathcal{G}} := m^*$. We first prove that $j < m_{\mathcal{G}}$.

By Proposition 4.4 we have $\alpha_{i,\mathcal{F}} = \alpha_{i,\mathcal{G}} := \alpha_i$, $h_{i,\mathcal{F}} = h_{i,\mathcal{G}} := h_i$, $k_{i,\mathcal{F}} = k_{i,\mathcal{G}} := k_i$ and $l_{i,\mathcal{F}} = l_{i,\mathcal{G}} := l_i$ for any $0 \leq i \leq j$. By Proposition 4.2,

$$\frac{\mathrm{mult}(p_{\mathcal{F}}, c)}{d_{\mathcal{F}}} = \frac{\mathrm{mult}(p_{h_j,\mathcal{F}}, c)}{d_{h_j,\mathcal{F}}}.$$

Since κ is not necessarily suitable for the Puiseux series $h_j = 0$, so for $d_{\mathcal{G}}$ we may use Statement 3.9, but for $d_{h_j,\mathcal{G}}$ we need Statement 3.11. By using the above formula and the two statements we obtain

$$d_{h_j,\mathcal{G}} \geq d_{h_j,\mathcal{F}} - \frac{\operatorname{mult}(p_{h_j,\mathcal{F}}, c)}{\kappa} = \frac{d_{h_j,\mathcal{F}}}{d_{\mathcal{F}}}(d_{\mathcal{F}} - \frac{\operatorname{mult}(p_{\mathcal{F}}, c)}{\kappa}) = \frac{d_{h_j,\mathcal{F}}}{d_{\mathcal{F}}} d_{\mathcal{G}}.$$

The above formula and (10) gives

$$d_{\mathcal{G}} + d_{h_j,\mathcal{G}} \geq \frac{d_{\mathcal{G}}}{d_{\mathcal{F}}}(d_{\mathcal{F}} + d_{h_j,\mathcal{F}}) > \frac{d_{\mathcal{G}}}{d_{\mathcal{F}}}(\alpha_j d_{\mathcal{F}} + 1 - \pi(\mathcal{F})) =$$
$$\alpha_j d_{\mathcal{G}} + \frac{d_{\mathcal{G}}}{d_{\mathcal{F}}}(1 - \pi(\mathcal{F})) \geq \alpha_j d_{\mathcal{G}} + 1 - \pi(\mathcal{F}) - \frac{1}{\kappa} = \alpha_j d_{\mathcal{G}} + 1 - \pi(\mathcal{G}),$$

and by (10) this is equivalent to $j < m_{\mathcal{G}}$. By Proposition 4.4, the implication $j < m_{\mathcal{F}} \Rightarrow j < m_{\mathcal{G}}$ is equivalent with $\mathcal{F} \preceq \mathcal{G}$.

Assume now (12) holds. Suppose indirectly that $m = m^*$. By Proposition 4.2, Statement 3.11 and (13), by a similar way as above, we obtain

$$\mu d_{\mathcal{G}} + 1 - \pi(\mathcal{G}) = d_{\mathcal{G}} + d_{h_m,\mathcal{G}} \geq d_{\mathcal{F}} - \tfrac{\operatorname{mult}(p_{\mathcal{F}},c)}{\kappa} + d_{h_m,\mathcal{F}} - \tfrac{\operatorname{mult}(p_{h_m,\mathcal{F}},c)}{\kappa} =$$
$$d_{\mathcal{F}}(1 - \tfrac{\operatorname{mult}(p_{\mathcal{F}},c)}{\kappa d_{\mathcal{F}}}) + d_{h,\mathcal{F}}(1 - \tfrac{\operatorname{mult}(p_{h,\mathcal{F}},c)}{\kappa d_{h,\mathcal{F}}}) = (d_{\mathcal{F}} + d_{h,\mathcal{F}})\tfrac{d_{\mathcal{G}}}{d_{\mathcal{F}}} = (\mu d_{\mathcal{F}} + 1 - \pi(\mathcal{F}))\tfrac{d_{\mathcal{G}}}{d_{\mathcal{F}}} =$$
$$\mu d_{\mathcal{G}} + 1 - \pi(\mathcal{F}) - \tfrac{(1-\pi(\mathcal{F}))\deg(p_{\mathcal{G}})}{\kappa d_{\mathcal{F}}} > \mu d_{\mathcal{G}} + 1 - \pi(\mathcal{F}) - \tfrac{1}{\kappa} = \mu d_{\mathcal{G}} + 1 - \pi(\mathcal{G}),$$

which is a contradiction. $\square$

Proposition 6.4. *Set $\mathcal{F} \in T_a^+$ and $h := h_{\mathcal{F}}$. Then $\mathcal{F} \in T_a^{\searrow}$ if and only if $\deg(p_{h,\mathcal{F}}) \leq \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \deg(p_{\mathcal{F}})$.*

Moreover, if $\deg(p_{h,\mathcal{F}}) = \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \deg(p_{\mathcal{F}})$, then $\mathcal{F} \prec \mathcal{F}'$ for any suitable $\kappa \in \mathbb{N}^*$.

Proof. Set $p = p_{\mathcal{F}}$, $q = p_{h,\mathcal{F}}$, $\mu = \mu_{\mathcal{F}}$. From Proposition 4.6, one has $d_{\mathcal{F}} p q' - d_{h,\mathcal{F}} p' q = c p^{\mu}$. Moreover, $\deg(p_{h,\mathcal{F}}) \leq \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \deg(p_{\mathcal{F}})$ is equivalent with

$$(*) \qquad \deg(p) + \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \deg(p) - 1 > \mu \deg(p).$$

From Proposition 4.2 and (10) we have $d_{\mathcal{F}} + d_{h,\mathcal{F}} = \mu d_{\mathcal{F}} + 1 - \pi(\mathcal{F})$. Using this, we obtain, that $(*)$ is equivalent with

$$(**) \qquad \deg(p) + \frac{(\mu - 1) d_{\mathcal{F}} + 1 - \pi(\mathcal{F})}{d_{\mathcal{F}}} \deg(p) - 1 > \mu \deg(p).$$

Elementary calculation shows, that $(**)$ is equivalent with

$$\frac{(1-\pi(\mathcal{F}))\deg(p)}{d_{\mathcal{F}}} > 1,$$

which is evidently equivalent with $\mathcal{F} \in T_a^{\searrow}$.

Assume now that $\deg(p_{h,\mathcal{F}}) = \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}}\deg(p_{\mathcal{F}})$.

By using Statement 3.11, our assumption, Statement 3.9, our assumption again, (10), and that $\mathcal{F} \in T_a^{\searrow}$, in the next line of arguments we obtain

$$d_{\mathcal{F}'} + d_{h,\mathcal{F}'} \geq d_{\mathcal{F}} + \frac{\deg(p_{\mathcal{F}})}{\kappa} + d_{h,\mathcal{F}} + \frac{\deg(p_{h,\mathcal{F}})}{\kappa} = (d_{\mathcal{F}} + d_{h,\mathcal{F}})(1 + \frac{\deg(p_{\mathcal{F}})}{\kappa d_{\mathcal{F}}}) =$$
$$(d_{\mathcal{F}} + d_{h,\mathcal{F}})\frac{d_{\mathcal{F}'}}{d_{\mathcal{F}}} = (\alpha_m d_{\mathcal{F}} + 1 - \pi(\mathcal{F}))\frac{d_{\mathcal{F}'}}{d_{\mathcal{F}}} = \alpha_m d_{\mathcal{F}'} + 1 - \pi(\mathcal{F}) + \frac{(1-\pi(\mathcal{F}))\deg(p_{\mathcal{F}})}{\kappa d_{\mathcal{F}}} >$$
$$\alpha_m d_{\mathcal{F}'} + 1 - \pi(\mathcal{F}) + \frac{1}{\kappa} = \alpha_m d_{\mathcal{F}'} + 1 - \pi(\mathcal{F}').$$

This shows $J(f^+_{\mathcal{F}'}, h^+_{\mathcal{F}'}) = 0$, so $\mathcal{F} \prec \mathcal{F}'$. □

Corollary 6.1. *Set* $\mathcal{F} \in T_a^{\searrow} \cap (V_a) \setminus \{(0,y)\}$. *Then* $\deg(p_{h,\mathcal{F}}) = \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}}\deg(p_{\mathcal{F}})$, *therefore* $\mathcal{F} \prec \mathcal{F}'$ *for any suitable* $\kappa \in \mathbb{N}^*$.

Proof. Simplify the notations by $p := p_{\mathcal{F}}$, $q := p_{h,\mathcal{F}}$, $\mu := \mu_{\mathcal{F}}$. By Proposition 6.4 we only have to prove that $\deg(q) = \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}}\deg(p)$. Assume indirectly $\deg(q) \neq \frac{d_{f,\mathcal{F}}}{d_{\mathcal{F}}}\deg(p)$. Then from Proposition 4.6, one has

$$\deg(q) = (\mu - 1)\deg(p) + 1.$$

From Proposition 6.7 we have that for any root of p, say c one has $\mathrm{mult}(q,c) = (\mu-1)\mathrm{mult}(p) + 1$. Therefore,

$$(\mu-1)\deg(p)+1 = \deg(q) \geq \sum_{c:p(c)=0}((\mu-1)\mathrm{mult}(p,c)+1) = (\mu-1)\deg p + k,$$

where k is the number of ythe different roots of p. By the above inequality, $k = 1$, so $\mathcal{F} \notin V_a$. □

By Proposition 6.4 we can prove a statement similar to Proposition 6.3.

Proposition 6.5. $(0,y) \preceq (0,x)$.

Proof. Set $h = h_{(0,x)}$. By Proposition 6.4 one has $\deg(p_{h,(0,x)}) > \frac{d_{h,(0,x)}}{d_{(0,x)}}\deg(p_{(0,x)})$. By Statement 3.12, $\deg(p_{h,(0,x)}) \leq d_{h,(0,y)}$, $d_{h,(0,x)} \geq \deg(p_{h,(0,y)})$ $d_{(0,x)} = \deg(p_{(0,y)})$ and $\deg(p_{(0,x)}) = d_{(0,y)}$. By using these formulae for our inequality we obtain

$$d_{h,(0,y)} > \frac{\deg(p_{h,(0,y)})}{\deg(p_{(0,y)})}d_{(0,y)},$$

which shows that $(h^+_{(0,y)})^k \neq \ominus(f^+_{(0,y)})^l$ for any $k,l \in \mathbb{N}^*$, so $(0,x) \prec (0,y)$ cannot be true. Via Proposition4.5, this transforms into $(0,y) \preceq (0,x)$. □

Proposition 6.6. *Set $\mathcal{F} \in T_a^{\nearrow}$. Assume $\mathcal{F} = I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$, $u \in \mathbb{Q}^+$. Then for any $v \in Q^+$ with*

(i) $v \geq u$

(ii) $\mathcal{G} := I_P(v) \in T_a^+$

we have $\mathcal{G} \in T_a^{\nearrow}$.

Proof. In the case $v \geq 1$,

$$(1-v)\deg(p_{\mathcal{G}}) \leq 0 < d_{\mathcal{G}},$$

so $\mathcal{G} \in T_a^{\nearrow}$.

Consider the case $v < 1$. By our assumptions,

$$(1-u)\deg(p_{\mathcal{F}}) < d_{\mathcal{F}}.$$

Therefore from Statement 3.10, one has

$$(1-v)\deg(p_{\mathcal{G}}) \leq (1-v)\deg(p_{\mathcal{F}}) < d_{\mathcal{F}} - (v-u)\deg(p_{\mathcal{F}}) \leq d_{\mathcal{G}},$$

showing $\mathcal{G} \in T_a^{\nearrow}$. □

Proposition 6.7. *Let $\mathcal{F} \in T_a^{\searrow}$ and $\kappa \in \mathbb{N}^*$ be suitable for fg. Assume* $\deg(p_{\mathcal{F}}) > 1$. *Then for any $c \in \mathbb{C}$ such that $\mathcal{F} * c$ exists, $\mathcal{F} * c \in T_a^+$. Moreover, there exists $c \in \mathbb{C}$ such that $\mathcal{F} * c \in T_a^{\searrow}$.*

Proof. Assume $\mathcal{F} := I_P(u)$. Set $h = h_{\mathcal{F}}$, $p = p_{\mathcal{F}}$, $q = p_{h,\mathcal{F}}$.

First prove that if $\mathcal{F} * c := \mathcal{G}$ exists, then $\mathcal{G} \in T_a^+$. We have two possibilities. First assume that $h = g$. In this case we state that $\mathcal{F} = \mathcal{F}_P^* := I_P(v)$. By the definition, $v \leq u$. Therefore $\mathcal{F}_P^* \in T_{a,\text{pole}}$. From (v) of Proposition 5.3, by Statements 3.9 and 3.10, for any $w \in \mathbb{Q}^+$ with $w > v$, for $\mathcal{H} := I_P(w)$, $deg(p_{\mathcal{H}}) \leq 1$, so $u = v$ and $\mathcal{F} = \mathcal{F}_P^* \in T_{a,\text{pole}}$.

By Statement 3.9 and (v) of Proposition 5.3, one has $d_{\mathcal{G}} = d_{\mathcal{F}} - \frac{1}{\kappa}$, so we have to prove $d_{\mathcal{F}} \neq \frac{1}{\kappa}$. By Statement 3.8, $\kappa d_{\mathcal{F}}$, $d_{g,\mathcal{F}} \in \mathbb{Z}$. By (viii) of Proposition 5.3 and Lemma 2.1, $\kappa d_{\mathcal{F}} \neq 1$.

Assume $g \neq h$. Then $(g_{\mathcal{F}}^+)^l = \ominus f_{\mathcal{F}}^k$ for some $k, l \in \mathbb{N}^*$. Therefore, from Statement 3.9, one has $d_{g\mathcal{G}} = \frac{k}{l} d_{\mathcal{G}}$. Since $\mathcal{F} \in T_a^{\searrow}$, one gets $u < 1$ from the definition. By Statement 3.8, $u \leq 1 - \frac{1}{\kappa}$.

By Proposition 4.1,

$$\frac{k}{l} d_{\mathcal{G}} + d_{\mathcal{G}} = d_{g,\mathcal{G}} + d_{\mathcal{G}} \geq 1 - \pi(\mathcal{G}) = 1 - u - \frac{1}{\kappa} \geq 0.$$

Therefore we only need $d_{\mathcal{G}} \neq 0$. Since in this case, in the inequality above, equality must hold, we would obtain $d_{\mathcal{G}} = d_{g,\mathcal{G}} = 0$, and from Proposition 4.1, one has $J(f_{\mathcal{G}}^{+} g_{\mathcal{G}})^{+} = \ominus \xi^{-1}$, which is evidently a contradiction, since in this case the above Jacobian determinant is 0.

Now we prove the second part of our proposition. Proposition 6.4 says that $\deg(q) \leq \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \deg(p)$.

Since $p^{d_{h,\mathcal{F}}} \neq \ominus q^{d_{\mathcal{F}}}$, there exists $c \in \mathbb{C}$ with $\mathrm{mult}(q, c) < \frac{d_{h,\mathcal{F}}}{d_{\mathcal{F}}} \mathrm{mult}(p, c)$. By Statements 3.16 and 3.18, we may assume that $\mathcal{G} := \mathcal{F} * c$ exists.

We claim that $\mathcal{G} \in T_a^{\searrow}$.

Assume indirectly $\mathcal{G} =\in T_a^{\nearrow}$. Since (13) cannot hold, (14) holds, so by Proposition 6.3 $h = h_{\mathcal{G}}$. By our indirect assumption $\deg(p_{h,\mathcal{G}}) > \frac{d_{h,\mathcal{G}}}{d_{\mathcal{G}}} \deg(p_{\mathcal{G}})$. By Statement 3.9, this means $\mathrm{mult}(q, c) > \frac{d_{h,\mathcal{F}} - \kappa \mathrm{mult}(q,c)}{d_{\mathcal{F}} - \kappa \mathrm{mult}(p,c)} \mathrm{mult}(p, c)$, so $\mathrm{mult}(q, c)(\kappa d_{\mathcal{F}} - \mathrm{mult}(p, c)) > (\kappa d_{h,\mathcal{F}} - \mathrm{mult}(q, c)) \mathrm{mult}(p, c)$. Therefore $\mathrm{mult}(q, c) d_{\mathcal{F}} > d_{h,\mathcal{F}} \mathrm{mult}(p, c)$, contradicting the definition of c. □

Lemma 6.1. *Set* $\mathcal{F} =\in T_a^{+}$, $u := \pi(\mathcal{F})$. *Assume that* $\deg(p_{\mathcal{F}}) = 1$. *Then* $J(f_{\mathcal{F}}^{+}, g_{\mathcal{F}}^{+}) = \ominus t^{-u}$.

Proof. Set $\mathcal{F} := I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$. Assume that $\kappa \in \mathbb{N}^*$ is suitable for fg, and $\kappa u \in \mathbb{N}$. For any $n \in \mathbb{N}$ with $n \leq \kappa u$ define $\mathcal{F}_n := I_P(\frac{n}{\kappa})$. Evidently

$$\deg(p_{\mathcal{F}_n}) = \frac{k_f}{k_g} \deg(p_{g,\mathcal{F}_n}) \tag{21}$$

for $n = 0$. Let $n \leq \kappa u$ be the largest number with (21). $n \neq u\kappa$, by (iv) of Lemma 2.1, .

We state that $p_{\mathcal{F}_n}$ and $p_{g,\mathcal{F},n}$ do not have common roots. Otherwise, by Proposition 4.2, $(g_{\mathcal{F}_n}^{+})^{k_f} = \ominus (f_{\mathcal{F}_n}^{+})^{k_g}$, which contradicts the maximality of n, by Statement 3.9.

Therefore, by Proposition 4.1, $J(f_{\mathcal{F}_n}^{+}, g_{\mathcal{F}_n}^{+}) = \ominus$. Using Statement 3.9, by induction we obtain that $\deg(p_{g,\mathcal{F}_j}) = 0$ for any $n < j \leq \kappa u$. Therefore, $J(f_{\mathcal{F}}^{+}, g_{\mathcal{F}}^{+}) \neq 0$, so, by Proposition 4.1, $J(f_{\mathcal{F}}^{+}, g_{\mathcal{F}}^{+}) = \ominus$. □

Proposition 6.8. *Set* $\mathcal{F} \in T_a^{\searrow}$. *Assume* $\deg(p_{\mathcal{F}}) \neq 1$.

Then there exist $P \in \overline{R}_a \setminus R_a$, $u, v \in \mathbb{Q}^{+}$ *with the following properties.*

(i) $\mathcal{F} = I_P(u)$;

(ii) $v \geq u$;

(iii) $I_P(v) \in T_{a,\mathrm{pole}}$.

Proof. Let $u := \pi(\mathcal{F})$, $\kappa \in \mathbb{N}^*$ be suitable, with $\kappa u \in \mathbb{N}$. We define by recursion a sequence $\mathcal{F}_n \in T_a^{\searrow}$ with the property $\mathcal{F}_{n+1} = \mathcal{F}_n * c_n$, in the following way.

Set $\mathcal{F}_0 := \mathcal{F}$. Assume $\mathcal{F}_n$ is already defined. There are two cases. If $\mathcal{F}_n \in T_{\text{apole}}$, the we finish the sequence.

Assume now $\mathcal{F}_n \notin T_{\text{apole}}$. We state that $\deg(p_{\mathcal{F}_n}) \neq 1$. Set $\mathcal{F}_n = I_Q(u+\frac{n}{\kappa})$. Assume indirectly $\deg(p_{\mathcal{F}_n}) = 1$. By Lemma 6.1, $m_{\mathcal{F}_n} = 0$, so $\mathcal{F}_Q^* \in T_{a,\text{pole}}$. By (iv) and (v) of Proposition 5.3, and Statemennt 3.9, we obtain that $\mathcal{F}_m \in T_{a,\text{pole}}$ for some $m < n$, which is a contradiction.

Therefore, $\deg(p_{\mathcal{F}}) > 1$. By Proposition 6.7, we obtain that there exists $c_n \in \mathbb{C}$ with $\mathcal{F}_{n+1} := \mathcal{F} * c_n \in T_a^{\searrow}$, so we can continue the sequence.

Since $d_{\mathcal{F}_n} \leq u - \frac{n}{\kappa}$, the sequence cannot be an infinite sequence, so we have to finish it. Set $\mathcal{F}_n \in T_{a,\text{pole}}$, and $\mathcal{F}_n := I_P(v)$ for some $P \in \overline{R}_a \setminus R_a$ and $v = u + \frac{n}{\kappa}$. Then the properties (i), (ii) and (iii) evidently hold. $\square$

7 The finite critical values of g

As we have seen, the non-constant meromorphic function $g : \overline{R}_a \to \mathbb{C} \cup \{\infty\}$ admits some critical values, and the preimages of these critical values form a subset of $\overline{R}_a \setminus R_a$. In this section we are going to describe the multiplicity data of these points using the language of the Eggers – Wall tree.

Notation 7.1. *Let $\kappa \in \mathbb{N}$ be suitable. Define*

$$T_{a,\text{cv}} := \{\mathcal{F} \in T_a^0 : \; d_{g,\mathcal{F}} = 0\}.$$

Statement 7.1. *Assume $\mathcal{F} \in T_{a,\text{cv}}$. Then $\pi(\mathcal{F}) > 1$.*

Proof. From the definition of $T_{a,\text{cv}}$ we obtain $J(f_{\mathcal{F}}^+, g_{\mathcal{F}}^+) = 0$. Therefore, from Proposition 4.1 one has $0 = d_{\mathcal{F}} + d_{g,\mathcal{F}} > 1 - \pi(\mathcal{F})$. $\square$

Statement 7.2. *Let $P \in R_a \setminus R_a$, $u \in \mathbb{Q}^+$ such that $\mathcal{F} := I_P(u) \in T_{a,\text{cv}}$. Then $\mathcal{F}^* \notin T_{a,\text{pole}}$.*

Statement 7.3. *Set $P \in \overline{R}_a \setminus R_a$. Assume that there exists $u \in \mathbb{Q}^+$ such that $I_P(u) \in T_a^{\nearrow}$. Then there exists $v \in \mathbb{Q}^+$ such that $I_P(v) \in T_{a,\text{cv}}$.*

Proposition 7.1. *Set $\mathcal{F} \in T_{a,\text{cv}}$, $p := p_{\mathcal{F}}$, $q := p_{g,\mathcal{F}}$, $\mathcal{G} := \mathcal{F}^\circ$. Then $\mathcal{G} := I_P(v) \in T_a^+$.*

Set $K_{\mathcal{G},0} := (k_0, \dots, k_{m-1})$, $L_{\mathcal{G},0} := (l_0, \dots, l_{m-1})$ and $S_{\mathcal{G},0} := (s_0, \dots, s_{m-1})$ (cf. Proposition 4.2). Define by recursion the finite sequence of polynomials r_j in the following way:

(i) $r_0 := q$*;*

(ii) $r_{j+1} := r_j^{k_j} - s_j p^{l_j}$ *for* $j = 0, 1, \ldots, m-2$*.*

Then $\deg(r_j) = \frac{l_j}{k_j}\deg(p)$ *for any* $0 \le j \le m-1$*.*

Proof. By Statement 3.11, $d_{\mathcal{G}} \ge \pi(\mathcal{F}) - \pi(\mathcal{G}) > 0$, so $\mathcal{G} \in T_a^+$.

As in Proposition 4.2, we introduce the finite sequence of polynomials h_j by

(i) $h_0 := g$;

(ii) $h_{j+1} := h_j^{k_j} - s_j f^{l_j}$ for $j = 0, 1, \ldots, m-2$.

Our Statement is equivalent with the property that for any $0 \le j \le m-1$ one has $d_{h_j,\mathcal{G}} = 0$.

Since $d_{\mathcal{F}} = 0$, for any j h_j is a polynomial of f, and g, one has $d_{h_j,\mathcal{F}} \le 0$. Let us prove $d_{h_j,\mathcal{F}} \ge 0$. Set $\mathcal{G} := I_P(v)$ for some $P \in R_a \setminus R_a$.

Choose some $w \in \mathbb{Q}$ such that $\pi(\mathcal{G}) < w < \pi(\mathcal{F})$ and choose some suitable $\kappa \in \mathbb{N}^*$ such that $\kappa w \in \mathbb{N}$. Then for $\mathcal{H} := I_P(w)$, there exists a unique $c \in \mathbb{C}$ such that $\mathcal{H} * c$ exists. Therefore, from Proposition 6.8 and Statement 7.2, $\mathcal{H} \in T_a^{\nearrow}$. From Proposition 6.3, one ha $d_{h_j,\mathcal{H}} \deg(p_{\mathcal{H}}) = \deg(p_{h_j,\mathcal{H}}) d_{\mathcal{H}}$. By Statement 3.9, $d_{\mathcal{H}} = \deg(p_{\mathcal{H}})(w - \pi(\mathcal{F}))$. Therefore

$$d_{h_j\mathcal{H}} = \frac{\deg(p_{h_j,\mathcal{H}})}{\deg(p_{\mathcal{H}})} d_{\mathcal{H}} = \deg(p_{h_j,\mathcal{H}})(w - \pi(\mathcal{F})).$$

From Statement 3.11 we obtain $d_{h_j,\mathcal{G}} \ge 0$. □

Proposition 7.2. *Let* $P \in \overline{R}_a \setminus R_a$*. Then* $g(P) \in \mathbb{C}$ *if and only if there exists* $u \in \mathbb{Q}^+$ *such that* $I_P(u) \in T_{a,\mathrm{cv}}$*.*

Notation 7.2. *Let* $P \in \overline{R}_a \setminus R_a$ *such that* $g(P) \in \mathbb{C}$*. Define* $\widehat{\mathcal{F}}_P := I_P(u) \in T_{a,\mathrm{cv}}$*.*

Proposition 7.3. *Set* $\mathcal{F} \in T_{a,\mathrm{cv}}$*,* $u := \pi(\mathcal{F})$*,* $p := p_{\mathcal{F}}$*,* $q := p_{g,\mathcal{F}}$*.*

Let $\kappa \in \mathbb{N}^*$ *be suitable. Choose some* $c \in \mathbb{C}$ *such that* $\mathcal{F} * c$ *exists. Then* $p(c) = a$*.*

Set

$$R_a^* := \{P \in \overline{R}_a \setminus R_a : \ I_P(u + \frac{1}{\kappa}) = \mathcal{F} * c\}.$$

Then the set R_a^* *does not depend on the choice of* κ*, and for any* $P \in R_a^*$ *one has* $g(P) = q(c)$*.*

We also have the following inequality:

$$\sum_{P \in R_a^*} \Lambda(P) \geq \kappa_{\mathcal{F}} \pi(\mathcal{F}) - \kappa_{\mathcal{F}}.$$

In the special case $\mathrm{mult}(p_{\mathcal{F}} - a, c) = 1$*, we have* $R_a^* = \{P\}$*, and* $\Lambda(P) = \kappa_{\mathcal{F}} \pi(\mathcal{F}) - \kappa_{\mathcal{F}}$*.*

Proof. The existence of P and the formula for $g(P)$ is trivial. Now assume that $\mathrm{mult}(p_{\mathcal{F}} - a, c) = 1$. From Statements 3.9 and 3.18, we have that $R_a^* = \{P\}$ for some $P \in R_a \setminus R_a$. Evidently $\mathcal{F} = I_P(u)$ for $u = \pi(\mathcal{F})$.

Set $\mathcal{G} := \mathcal{F}_P^*$. Assume that $\mathcal{G} = I_P(v)$ for some $v \in \mathbb{Q}^+$. Then $v \geq u + \frac{1}{\kappa}$. Define $b := q(c)$. Then for any $w \in (u, v) \cap \mathbb{Q}$ we have for $\mathcal{H} := I_P(w)$:

(i) $\mathcal{H} \in T_a^-$;

(ii) $J((f-a)_{\mathcal{H}}^+, (g-b)_{\mathcal{H}}^+) = 0$;

(iii) $\deg(p_{f-a,\mathcal{H}}) = 1$.

From Propositions 4.3, 4.4 and (iii), we have that $(g_{\mathcal{H}} - b)^+ = \ominus((f-a)_{\mathcal{F}}^+)^k$ for $k = \mathrm{mult}(q, c)$. Therefore, from Statement 3.11, $d_{g-b,\mathcal{G}} = k d_{f-a,\mathcal{G}}$. From Proposition 4.1 and from Statement 3.9 one gets

$$1 - v = d_{f-a,\mathcal{G}} + d_{g-b,\mathcal{G}} = (k+1) d_{f-a,\mathcal{G}} = (k+1)(u - v),$$

consequently $v = \frac{(k+1)u-1}{k}$. Hence, by Statements 3.15, one has

$$\Lambda(P) = -D_{g-b,\mathcal{G}} = -\kappa_{\mathcal{F}} d_{g-b,\mathcal{G}} = \frac{\kappa_{\mathcal{F}}(v-1)}{k+1} = \kappa_{\mathcal{F}} \frac{(k+1)(u-1)}{k+1} = \kappa_{\mathcal{F}} u - \kappa_{\mathcal{F}}.$$

Next we prove the inequality. We give a geometric proof.

Chose some $\rho > 0$ such that for any $c^* \in \dot{B}(c, 2\rho)$ one has $p_{\mathcal{F}}(c^*) \neq 0$. For any $a^* \in \mathbb{C}$ and for $\mathcal{G} = M_{a,a^*}$ (cf. Statement 3.14) one has

$$p_{\mathcal{G}} = p_{\mathcal{F}} \quad \text{and} \quad p_{g,\mathcal{G}} = p_{g,\mathcal{F}}.$$

Therefore, if $a^* \in \mathbb{C}$ is generic, then $p_{\mathcal{G}}$ is squarefree. Moreover, if $\varepsilon > 0$ and $\delta > 0$ are small enough, then for any $a^* \in B(a, \delta)$ and for $\mathcal{G} = M_{a,a^*}$ we have the following properties:

(i) For any $c^* \in \dot{B}(c, \rho)$ one has $p_{\mathcal{G}}(c^*) \neq 0$;

(ii) $\overline{g^{-1}(B(b, \varepsilon))} \cap (\overline{R}_a \setminus R_a) \subset g^{-1}(b)$;

(iii) $g^{-1}(C(b, \varepsilon)) \cap (R_{a^*} \setminus R_{a^*} = \emptyset$.

Let A be the union of the components of $g^{-1}(B(b,\varepsilon))$ which contains some $P \in R_a^*$. Define the set $A_{a^*} := A \cap R_a^*$ for any $a^* \in B(a,\delta)$.

By its definition, any point in $B(b,\varepsilon)$ has exactly $\sum_{P\in R_a^*} \Lambda(P)$ preimages in $B(a,\delta)$. Chose any $a^* \in B(a,\delta)$. Then there exists $Q \in (\overline{R}_{a^*} \setminus R_{a^*}) \cap A_{a^*}$. Moreover, $I_Q(u) = M_{a,a^*}(\mathcal{F}) := \mathcal{G}$. Assume further that $p_{\mathcal{G}}$ is squarefree (which holds for generic a^*). Then

$$\textstyle\sum_{P\in R_a^*} \Lambda(P) \geq \Lambda(Q) = \kappa_{\mathcal{G}}\pi(\mathcal{G}) - \kappa_{\mathcal{G}} = \kappa_{\mathcal{F}}\pi(\mathcal{F}) - \kappa_{\mathcal{F}}. \quad \square$$

Notation 7.3. *Define*

$$\delta_a := \sum_{P\in \overline{R}_a \setminus R_a \ :g(P)\in\mathbb{C}} (\Lambda(P) - \kappa_{\widehat{\mathcal{F}}_P}(\pi(\widehat{\mathcal{F}}_P - 1))).$$

Proposition 7.4. *For any $a \in \mathbb{C}$ one has $\delta_a \geq 0$. Moreover, for generic a we have $\delta_a = 0$.*

Proposition 7.5. *Chose any $a_0 \in \mathbb{C}$. Assume that $T_{a_0,\mathrm{cv}} = \{\mathcal{F}_1, \ldots, \mathcal{F}_s\}$.*
Then

$$td(f,g) = 1 + \sum_{i=1}^{s} \kappa_{\mathcal{F}_i}(\pi(\mathcal{F}_i) - 1) + \sum_{a\in\mathbb{C}} \delta_a. \tag{22}$$

Proof. We first remark that each generic point has $td(f,g)$ preimages. More explicitly, any point in the set $\mathbb{C}^2 \setminus \big\{\big(p_{\mathcal{F}_i}(c), p_{g,\mathcal{F}_i}(c)\big) : \ c \in \mathbb{C}, 1 \leq i \leq s\big\}$, has $td(f,g)$ preimages in $\mathbb{C}^2$.

Introduce the following notation. Set $(a,b) \in \mathbb{C}^2$. Denote $(a \to_i b)$ if there exists $c \in \mathbb{C}$ with

(i) $p_{\mathcal{F}_i}(c) = a$.

(ii) $p_{g,\mathcal{F}_i}(c) = b$.

By Proposition 7.3, and from Statement 3.14, for generic $a \in \mathbb{C}$, the point (a,b) has

$$td(f,g) - \sum_{i:\ a\to_i b} \big(\pi(\mathcal{F}_i)\kappa_{\mathcal{F}_i} - \kappa_{\mathcal{F}_i}\big)$$

preimages. Moreover, for any $a \in \mathbb{C}$,

$$\sum_{b\in\mathbb{C}} \Big(td(f,g) - \sum_{i:\ a\to_i b} \big(\pi(\mathcal{F}_i) - 1\big)\kappa_{\mathcal{F}_i} - \#(f,g)^{-1}\big((a,b)\big)\Big) = \delta_a.$$

Since the set of points $(p_{\mathcal{F}_i}(c), p_{g,\mathcal{F}_i}(c))$ is biholomorphic with $\mathbb{C}$, via Notation 7.3 and by calculating the Euler characteristics of $\mathbb{C}^2$ via its subsets, we obtain

$$1 = td(f,g) - \sum_{i=1}^{s} \Big(\big(\pi(\mathcal{F}_i) - 1\big)\kappa_{\mathcal{F}_i} \Big) + \sum_{a\in\mathbb{C}} \delta_{\mathcal{F}_i,a},$$

which is equivalent with (22). $\square$

Corollary 7.1. *Set $a \in \mathbb{C}$ and $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\} \subset T_{a,\mathrm{cv}}$. Then*

$$td(f,g) \geq 1 + \sum_{i=1}^{n} \kappa_{\mathcal{F}_i}(\pi(\mathcal{F}_i) - 1).$$

Proof. From Propositions 7.5, 7.4, and Statement 7.1 one has

$td(f,g) \geq 1 + \sum_{i=1}^{n} \kappa_{\mathcal{F}_i}(\pi(\mathcal{F}_i) - 1) + \sum_{a\in\mathbb{C}} \delta_a \geq 1 + \sum_{i=1}^{n} \kappa_{\mathcal{F}_i}(\pi(\mathcal{F}_i) - 1).$ $\square$

8 Some number theoretical properties

In this section we prove some number theoretical properties of the elements of T_a.

Notation 8.1. *Set $\mathcal{F} \in T_a$, $m = m_{\mathcal{F}}$, $h_i = h_{i,\mathcal{F}}$ $(i = 0, \ldots, m)$ (cf Proposition 4.2). Define*

$$M_{\mathcal{F}} := \gcd.(\deg p_{\mathcal{F}}, \deg p_{h_0,\mathcal{F}}, \ldots, \deg p_{h_m,\mathcal{F}}),$$

$$M^*_{\mathcal{F}} := \gcd.(\deg p_{\mathcal{F}}, \deg p_{h_0,\mathcal{F}}, \ldots, \deg p_{h_{m-1},\mathcal{F}}).$$

By using elementary number theory, we obtain useful formula for $M_{\mathcal{F}}$.

Statement 8.1. *Set $\mathcal{F} \in V_a$. Set $m = m_{\mathcal{F}}$ and $h_j := h_{j,\mathcal{F}}$ $(j = 0, \ldots, m)$. Then there exist N, $N_0, \ldots, N_m, N^*, N^*_0, \ldots, N^*_{m-1} \in \mathbb{Z}$ with*

$$M_{\mathcal{F}} = N \deg(p_{\mathcal{F}}) + N_0 \deg(p_{h_{0,\mathcal{F}}}) + \cdots + N_m \deg(p_{h_{m,\mathcal{F}}}),$$

and

$$M^*_{\mathcal{F}} = N^* \deg(p_{\mathcal{F}}) + N^*_0 \deg(p_{h_{0,\mathcal{F}}}) + \cdots + N^*_{m-1} \deg(p_{h_{m-1,\mathcal{F}}}).$$

Proposition 8.1. *Set $\mathcal{F} =\in T_a^{\searrow}$, $u := \pi(\mathcal{F})$ and*

$$i := \frac{\deg(p_{\mathcal{F}})}{M^*_{\mathcal{F}}}.$$

Then $i \in \mathbb{N}^*$ and we have the following properties:

(i) There exists a polynomial p and $\delta \in \mathbb{Q}$ such that $(\xi^\delta p(\eta))^i = \ominus f^+_{\mathcal{F}}(\xi,\eta)$;

(ii) There exists polynomial q with $h^+_{\mathcal{F}}(\xi,\eta) = \xi^{1-u}(\xi^\delta p(\eta))^k q(\eta)$, where $k = i(\mu_{\mathcal{F}} - 1) \in \mathbb{Z}$;

(iii) $J(\xi^\delta p(\eta), \xi^{1-u} q(\eta)) = \ominus \xi^{\delta-u} p(\eta)$;

(iv) $\delta p q' - (1-u) p' q = \ominus p$;

(v) $M_{\mathcal{F}} = \gcd .(\deg(p), \deg(q))$.

Proof. Property (i) follows from the definition of $M^*_{\mathcal{F}}$.

Via the notations of Statement 8.1, define

$$\xi^\delta p(\eta) := ((f^+_{\mathcal{F}})^{N^*}(h^+_{0,\mathcal{F}})^{N^*_0} \ldots (h^+_{m-1,\mathcal{F}})^{N^*_{m-1}})(\xi,\eta).$$

By Proposition 4.2, $(\xi^\delta p(\eta)) = \ominus (f^+_{\mathcal{F}}(\xi,\eta))^j$ for some $j \in \mathbb{Q}$. From Statement 8.1, one has $j = \frac{1}{i}$, which proves property (ii).

Set $0 \le j \le m-1$. Since

$$\deg(p) = \frac{\deg(p_{\mathcal{F}})}{i} = M^*_{\mathcal{F}} \mid \deg(p_{h_j,\mathcal{F}}) = \frac{l_j}{k_j} \deg(p_{\mathcal{F}})$$

(cf. Proposition 4.2), we obtain $i\frac{l_j}{k_j} \in \mathbb{N}$, so $\frac{i}{k_j} \in \mathbb{N}$. This, however gives $i\mu_{\mathcal{G}} \in \mathbb{N}$, so $k \in \mathbb{Z}$.

Define

$$H(\xi,\eta) := \frac{h^+_{\mathcal{F}}(\xi,\eta)}{(\xi^\delta p(\eta))^k}.$$

By its definition, $H(\xi,u) = \xi^l q(\eta)$ for some rational function q and $l \in \mathbb{Z}$. By applying Proposition 4.2, for the constant l one has

$$l = d_{h,\mathcal{F}} - k\delta = (\mu_{\mathcal{F}} - 1) d_{\mathcal{F}} + 1 - u - k\delta = (i\mu_{\mathcal{F}}\delta - i\delta) + 1 - u - (i\mu_{\mathcal{F}} - 1)\delta = 1 - u.$$

Moreover,

$$q = \frac{p_{h,\mathcal{F}}}{p^k} = \frac{p_{h,\mathcal{F}}}{p^{\mu-1}_{\mathcal{F}}}.$$

Therefore, from Proposition 6.3, we obtain that for any root of $p_{\mathcal{F}} c$, one has $\mathrm{mult}(p_{h\mathcal{F}}, c) - (\mu - 1)\mathrm{mult}(p_{\mathcal{F}}) = 1$, which implies that q is a polynomial, and $H(\xi,\eta) = \xi^{u-1} q(\eta)$.

By the chain rule

$$\ominus \xi^{-u} \big(\xi^\delta p(\eta)\big)^{i\mu} = \ominus \xi^{-u} \big(f^+_{\mathcal{F}}(\xi,\eta)\big)^{\mu} = J\big(f^+_{\mathcal{F}}(\xi,\eta), h^+_{\mathcal{F}}(\xi,\eta)\big) = J\big((\xi^\delta p(\eta))^i, (\xi^\delta p(\eta))^k H(\xi,\eta)\big) =$$
$$\ominus \big(\xi^\delta p(\eta)\big)^{i-1} \big(\xi^\delta p(\eta)\big)^k J\big(\xi^\delta p(\eta), H(\xi,\eta)\big) = \ominus \big(\xi^\delta p(\eta)\big)^{i\mu_{\mathcal{F}}-1} J\big(\xi^\delta p(\eta), H(\xi,\eta)\big),$$

so $J(\xi^\delta p(\eta), H(\xi,\eta)) = \ominus\xi^{\delta-u}p(\eta)$. This shows (iii). (iv) is a simple corollary of (iii).

As we noticed, $\deg(p) = M^*_{\mathcal{F}}$. Therefore,

$$M_{\mathcal{F}} = \gcd.(M_{\mathcal{F}}, \deg(p_{h_m,\mathcal{F}})) = \gcd.(M_{\mathcal{F}}, \deg(p_{h_m,\mathcal{F}}) - kM^*_{\mathcal{F}}) = gcd(\deg(p), \deg(q)).\square$$

Statement 8.2. *Use the notations of Proposition 8.1. Set $\mathcal{F} \in (T_a^{\searrow} \cap V_a) \setminus T_{a,\text{pole}}$, $\mathcal{G} := \mathcal{F} + c$ for some $c \in \mathbb{C}$. Then $\deg(q)\text{mult}(p,c) \neq \deg(p)$. Moreover, $\mathcal{G} \in T_a^{\searrow}$ if and only if $\deg(q)\text{mult}(p,c) > \deg(p)$.*

Proof. From statement 3.16, we have that p has more than one root. Therefore, by (iv) of Proposition 8.1 one has $\frac{\deg(p)}{\deg(q)} = \frac{\delta}{1-u}$.

Assume indirectly $\deg(q)\text{mult}(p,c) = \deg(p)$. From (iv) of Proposition 8.1 one has $\text{mult}(q,c) = 1$. Therefore,

$$\frac{\text{mult}(p,c)}{\text{mult}(q,c)} = \frac{\deg(p)}{\deg(q)} = \frac{\delta}{1-u}.$$

Therefore, $\text{mult}(p,c) = \text{mult}(\delta p q' - (1-u)p'q) > \text{mult}(p,c)$, which is a contradiction.

Set

$$W := d_{\mathcal{F}} - (1-\pi(\mathcal{F}))\text{mult}(p_{\mathcal{F}},c) = i\delta - (1-u)i\text{mult}(p,c) =$$
$$i(\delta - \tfrac{\delta\deg(q)\text{mult}(p,c)}{\deg(p)} = i\delta\tfrac{\deg(p)-\deg(q)\text{mult}(p,c)}{\deg(p)},$$

Therefore, by Statement 6.2 one has $\mathcal{G} \in T_a^{\searrow}$ if and only if $W < 0$ if and oly if $\deg(q)\text{mult}(p,c) > \deg(p)$. $\square$

Statement 8.3. *Let $\kappa \in \mathbb{N}^*$ be suitable. Set $\mathcal{F} \in T_a^{\searrow} \cap (V_a \setminus \{(0,y)\} := \mathcal{G} * c$. Then for any $0 \le j \le m_{\mathcal{F}}$ one has:*

(i) $h_{j,\mathcal{F}} = h_{j,\mathcal{G}} := h_j$;

(ii) $\deg(p_{h_i,\mathcal{F}}) = \text{mult}(p_{h_j,\mathcal{G}}, c)$.

Proof. From Statements 3.9 and 3.16 one has $\text{mult}(p_{\mathcal{G}},c) = \deg(p_{\mathcal{F}})$. From Corollary 6.1 and from Proposition 6.3 we obtain $\mathcal{F} \prec \mathcal{G}$. Use the notations of Proposition 4.2.

Evidently, for $0 \le j \le m_{\mathcal{F}} - 1$ one has $k_{j,\mathcal{F}} = k_{j,\mathcal{G}} := k_j$, $l_{j,\mathcal{F}} = l_{j,\mathcal{G}} := l_j$, $s_{j,\mathcal{F}} = s_{j,\mathcal{G}} := s_j$, and for any $0 \le j \le m_{\mathcal{F}}$, $h_{j,\mathcal{F}} = h_{j,\mathcal{G}} = h_j$, which provides (i).

Moreover, from (iii) of Proposition 4.2, for any $0 \le j \le m_{\mathcal{F}} - 1$ we obtain

$$\frac{\deg(p_{h_j,\mathcal{G}})}{\deg(p_{\mathcal{G}})} = \frac{\text{mult}(p_{h_j,\mathcal{G}},c)}{\text{mult}(p_{\mathcal{G}},c)} = \frac{k_j}{l_j} = \frac{\deg(p_{h_j,\mathcal{F}})}{\deg(p_{\mathcal{F}})},$$

hence

$$\deg(p_{h_j,\mathcal{F}}) = \frac{k_j}{l_j}\deg(p_{\mathcal{F}}) = \frac{k_j}{l_j}\mathrm{mult}(p_{\mathcal{G}}, c) = \mathrm{mult}(p_{h_j,\mathcal{G}}, c),$$

which proves (ii) for $0 \leq j \leq m_{\mathcal{F}} - 1$.

Finally we prove our statement for $i = m_{\mathcal{F}} := m$. From Corollary 6.1

$$\frac{d_{h_m,\mathcal{F}}}{d_{\mathcal{F}}} = \frac{\deg(p_{h_m,\mathcal{F}})}{\deg(p_{\mathcal{F}})} := \psi.$$

We state that

$$\psi = \frac{k_m}{l_m} := \frac{k_{m,\mathcal{G}}}{l_{m,\mathcal{G}}}.$$

Set $\mathcal{F} := I_P(u)$ for some $P \in \overline{R}_a \setminus R_a$, and $q \in \mathbb{Q}^+$. Then $\mathcal{G} = I_P(u - \frac{1}{\kappa})$. Chose $i \in \mathbb{N}^*$ with the property that $i\kappa$ is suitable for h_m. Set $\mathcal{H} := I_P(u - \frac{1}{i\kappa})$.

From Corollary 6.1 we obtain that $\mathcal{F} \prec \mathcal{H}$. From Proposition 6.3, one has $\mathcal{H} \preceq \mathcal{G}$, hence, by using Proposition 4.2, one has

$k_m = k_{m,\mathcal{G}} = k_{m,\mathcal{H}}$ and $l_m = l_{m,\mathcal{G}} = l_{m,\mathcal{H}}$. From Statement3.9, one has

$$\psi = \tfrac{d_{h_m,\mathcal{F}}}{d_{\mathcal{F}}} = \tfrac{i\kappa d_{h_m,\mathcal{F}} + \deg(p_{h_m,\mathcal{F}})}{i\kappa d_{\mathcal{F}} + \deg(p_{\mathcal{F}})} = \tfrac{d_{h_m,\mathcal{H}}}{d_{\mathcal{H}}} = \tfrac{k_{m,\mathcal{H}}}{l_{m,\mathcal{H}}} = \tfrac{k_{m,\mathcal{G}}}{l_{m,\mathcal{G}}}. \quad \square$$

Statement 8.4. *Set $\mathcal{F}, \mathcal{G} \in V_a \cap T_a^{\searrow}$. Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. Then, using the notations of Proposition 8.1,* $\mathrm{mult}(p, c) \mid M_{\mathcal{G}}$.

Proof. Set $m := m_{\mathcal{G}}$. From Statement 8.1 one has

$$M_{\mathcal{G}} = N\deg(p_{\mathcal{G}}) + N_0\deg(p_{h_{0,\mathcal{G}}}) + \cdots + N_m^*\deg(p_{h_m,\mathcal{G}}).$$

Set

$$\xi^v p^*(\eta) := ((f_{\mathcal{F}}^+)^N (h_{0,\mathcal{F}}^+)^{N_0} \dots (h_{m-1,\mathcal{F}}^+)^{N_{m-1}})(\xi, \eta).$$

By Proposition6.7, using the same argument as by the prrof of Proposition 8.1, we obtain, that p^* is a polynomial, and by Statement 8.3 one has

$$\mathrm{mult}(p^*, c) = N\mathrm{mult}(p_{\mathcal{F}}, c) + N_0\mathrm{mult}(p_{h_0,\mathcal{F}}, c) + \cdots + N_m\mathrm{mult}(p_{h_m,\mathcal{F}}, c) =$$
$$N\deg(p_{\mathcal{G}}) + N_0\deg(p_{h_0,\mathcal{G}}) + \cdots + N_m\deg(p_{h_m,\mathcal{F}}) = M_{\mathcal{G}}.$$

Moreover, by Proposition6.7, $p^* = p^i$, for some $i \in \mathbb{N}^*$. $\square$

Statement 8.5. *Set $P \in \overline{R}_a \setminus R_a$, $\mathcal{F} \in T_a^{\searrow} \cap V_a$, and $\mathcal{G} := \mathcal{F}^\circ$. Assume $\mathcal{G} \notin V_{2,a}$. Then $M_{\mathcal{G}} \mid M_{\mathcal{F}}$.*

Proof. Assume $\mathcal{F} = I_P(u)$ and $\mathcal{G} = I_P(v)$ for some $P \in \overline{R}_a \setminus R_a$ and for some $u, v \in \mathbb{Q}^+$. Evidently, $v < u$. Choose some $\kappa \in \mathbb{N}^*$ suitable such that $\kappa(u - v) \in \mathbb{N}^*$. Define $\mathcal{H} := I_P(u - \frac{1}{\kappa})$. Then $\mathcal{F} = \mathcal{H} * c^*$ for some $c^* \in \mathbb{C}$.

Let $\nu := \nu_{\mathcal{G}}$. From the assumption $\mathcal{G} \notin V_{2,a}$ and from Statement 3.18 one has $p_{\mathcal{G}}(\eta) = \ominus(\eta^\nu - c^\nu)^l$ for some $c \in \mathbb{C}^*$, and for some $l \in \mathbb{N}^*$. From Proposition 6.3 and Corollary 6.1 one has $\mathcal{F} \prec \mathcal{H} \preceq \mathcal{G}$. Therefore, $m := m_{\mathcal{F}} < m_{\mathcal{H}} \leq m_{\mathcal{G}}$. Moreover, from Statement 8.3 one has

$$\deg(p_{h_j,\mathcal{F}}) = \operatorname{mult}(p_{h_j,\mathcal{H}}, c^*) = \tfrac{k_j}{l_j}\operatorname{mult}(p_{\mathcal{H}}, c^*) = \tfrac{k_j}{l_j}\operatorname{mult}(p_{\mathcal{G}}, c) = \operatorname{mult}(p_{h_j,\mathcal{G}}, c) = \tfrac{\deg(p_{h_j,\mathcal{G}})}{\nu}$$

for any $0 \leq j \leq m$. Therefore,

$$M_{\mathcal{G}} \mid \gcd.(\deg(p_{\mathcal{G}}), \deg(p_{h_0,\mathcal{G}}), \ldots, \deg(p_{h_m,\mathcal{G}}) \deg(p_{h,\mathcal{G}})) = \gcd.(\nu \deg(p_{\mathcal{F}}), \nu \deg(p_{h_0,\mathcal{F}}), \ldots, \nu \deg(p_{h_m,\mathcal{F}}) \deg(p_{h,\mathcal{G}})) = \gcd.(\nu M_{\mathcal{F}}, \deg(p_{h,\mathcal{G}})).$$

In order to finish the proof we show that $\gcd.(\nu, \deg(p_{h,\mathcal{F}})) = 1$. Since $p_{\mathcal{G}}(\eta) = \widetilde{p}(\eta^\nu)$, for some polynomial $\widetilde{p}$, by Proposition 4.6, one gets $p_{h,\mathcal{G}}(\eta) = \eta r(\eta^\nu)$ for some polynomial r, which implies the required formula. $\square$

Proposition 8.2. *Set $\mathcal{F} \in T_a^{\searrow} \cap (V_a \setminus \{(0, y)\})$, and $\mathcal{G} := \mathcal{F}^\circ$. Set $M := M_{\mathcal{F}}$, $u := \pi(\mathcal{F})$ and $v := \pi(\mathcal{G})$.*

Assume

$$d_{\mathcal{F}} \leq \deg(p_{\mathcal{F}}) \left(\frac{1 - v}{M} + v - u\right).$$

Let $\kappa \in \mathbb{N}^*$ be suitable. Then for any $c \in \mathbb{C}$ such that $\mathcal{H} := \mathcal{G} * c$ exists, one has $\mathcal{H} \in T_a^{\searrow}$.

Proof. Set $P \in \overline{R}_a$ such that $\mathcal{F} = I_P(u)$. Then $\mathcal{G} = I_P(v)$. By our assumptions, $v < u < 1$.

Define $c^* \in \mathbb{C}$ by $\mathcal{G} * c^* = I_P(v + \frac{1}{\kappa})$. Then, from Statements 3.16 and 3.9, one has $\operatorname{mult}(p_{\mathcal{G}}, c^*) = \deg(p_{\mathcal{F}})$. Let $m := m_{\mathcal{F}}$. Chose $N, N_0, \ldots, N_m \in \mathbb{Z}$ such that

$$M_{\mathcal{F}} = N \deg(p_{\mathcal{F}}) + N_0 \deg(p_{h_{0,\mathcal{F}}}) + \cdots + N_m \deg(p_{h_{m,\mathcal{F}}})$$

(cf. Statement 8.1). From Proposition 6.3 and Corollary 6.1, one has $\mathcal{F} \prec \mathcal{G}$. Define

$$p := p_{\mathcal{G}}^N \prod_{j=1}^m p_{h_j,\mathcal{G}}^{N_j}.$$

By Proposition 4.2 and by $m < m_{\mathcal{G}}$ one has that $p = p^w$ for some $w \in \mathbb{Q}$, and p is a rational function. From Statement 8.3, one has $\mathrm{mult}(p, c^*) = M$, therefore p is a polynomial. Since $\mathrm{mult}(p, c) \geq 1$,

$$\mathrm{mult}(p_{\mathcal{G}}, c) \geq \frac{\mathrm{mult}(p_{\mathcal{G}}, c^*)}{M} = \frac{\deg(p_{\mathcal{F}})}{M}.$$

Therefore,

$$d_{\mathcal{H}} + (\pi(\mathcal{H}) - 1)\deg(p_{\mathcal{H}}) = d_{\mathcal{G}} - \tfrac{\mathrm{mult}(p_{\mathcal{G}}, c)}{\kappa} + (v + \tfrac{1}{\kappa} - 1)\mathrm{mult}(p_{\mathcal{G}}, c) =$$
$$d_{\mathcal{G}} + (v-1)\mathrm{mult}(p_{\mathcal{G}}, c) = d_{\mathcal{F}} + (u - v)\deg(p_{\mathcal{F}}) + (v-1)\mathrm{mult}(p_{\mathcal{G}}, c) \leq$$
$$d_{\mathcal{F}} + (u-v)\deg(p_{\mathcal{F}}) + \tfrac{v-1}{M}\mathrm{mult}(p_{\mathcal{G}}, c) \leq 0.$$

Therefore, $\mathcal{H} \notin T_a^{\nearrow}$, so by Statement 6.1, $\mathcal{H} \in T_a^{\searrow}$. □

Corollary 8.1. *Set $\mathcal{F} \in T_a^{\searrow} \cap (V_a \setminus \{(0, y)\})$, and $\mathcal{G} := \mathcal{F}^{\circ}$. Assume $M_{\mathcal{F}} = 1$. Then for any $c \in \mathbb{C}$ such that $\mathcal{G} * c$ exists, one has $\mathcal{G} * c \in T_a^{\searrow}$.*

Proposition 8.3. *Set $\mathcal{F} \in T_a^{\searrow} \cap (V_a \setminus \{(0, y)\})$, and $\mathcal{G} := \mathcal{F}^{\circ}$. Assume that for any $\mathcal{H} \in T_a^{\searrow} \cap (V_a \setminus \{(0, y)\})$ one has $\mathcal{G} \neq \mathcal{H}^{\circ}$, and assume $M_{\mathcal{F}} = 1$. Then we have the following properties:*

(i) $M_{\mathcal{G}} = 1$;

(ii) If $\mathcal{G} \neq (0, y)$, then $p_{\mathcal{G}}(\eta) = \ominus(\eta^{\nu} - c^{\nu})^M$, for some $c \in \mathbb{C}^*$, where $\nu = \nu_{\mathcal{G}} \neq 1$;

(iii) $p_{\mathcal{G}}(\eta) = (\eta - c)^k$ for some $k \in \mathbb{N}^*$ if $\mathcal{G} = (0, y)$.

Proof. By our assumptions and by Corollary 8.1, $\mathcal{G} \notin V_{2,a}$. Therefore (i) follows from Statement 8.5.

Now we prove (ii). By our assumptions $\mathcal{G} \in V_{1,a} \setminus V_{2,a}$. Therefore, from Statements 3.16 and 3.18, one has $\nu_{\mathcal{G}} \neq 1$, and (ii).

Finally, consider the case $\mathcal{G} = (0, y)$. Since $\mathcal{G} \notin V_{2,a}$, and since $\nu_{\mathcal{G}} = 1$, by Statement 3.16 we obtain (iii). □

Proposition 8.4. *Let (f, g) be a normalized counterexample of the Jacobian conjecture. Assume that $T_{a,\mathrm{pole}} = \{\mathcal{G}\}$. Then for any $\mathcal{F} \in T_a^{\searrow} \cap V_a$ one has $M_{\mathcal{F}} \neq 1$.*

Proof. Assume indirectly, that our statement does not hold. Define the finite sequence $\mathcal{F}_0, \ldots, \mathcal{F}_n$ in the following way:

(i) $\mathcal{F}_0 := \mathcal{F}$;

(ii) If $\mathcal{F}_j$ is defined, and $\pi(\mathcal{F}_j) \neq 0$, then set $\mathcal{F}_{n+1} = \mathcal{F}^\circ$. Otherwise we stop.

From Propositions 6.7 and 6.8, we obtain $\mathcal{F}_0, \ldots, \mathcal{F}_n \in T_a^{\searrow}$. Therefore $\mathcal{F}_n = (0, y)$. Set $\mathcal{H} := \mathcal{F}_{n-1}$.

From Proposition 8.3, and by induction we have $M_{\mathcal{H}} = 1$. Set $m = m_{\mathcal{H}}$. Set $N, N_0, \ldots, N_m \in \mathbb{Z}$ such that

$$1 = N \deg(p_{\mathcal{H}}) + N_0 \deg(p_{h_0,\mathcal{H}}) + \cdots + N_m \deg(p_{h_m,\mathcal{H}})$$

(cf. Statement 8.1). Set

$$(k, l) := N\big(\deg(p_{(0,y)}), d_{(0,y)}\big) + N_0\big(\deg(p_{h_0,(0,y)}), d_{h_0(0,y)}\big) + \cdots + N_m\big(\deg(p_{h_m,(0,y)}), d_{h_m(0,y)}\big).$$

From Corollary 6.1 and from Proposition 6.3, there exists $w \in \mathbb{Q}$ such that $(k, l) = w(k_f, l_f)$. Since $d_{h,(0,y)} \in \mathbb{N}$ for any polynomial $h(x, y)$ one gets $k, l \in \mathbb{Z}$. Moreover,

$$\begin{aligned} k &= N \deg(p_{(0,y)}) + N_0 \deg(p_{h_0,(0,y)}) + \cdots + N_m \deg(p_{h_m,(0,y)}) \\ &= N \mathrm{mult}(p_{(0,y)}, c) + N_0 \mathrm{mult}(p_{h_0,(0,y)}, c) + \cdots + N_m \mathrm{mult}(p_{h_m,(0,y)}, c) = \\ &= N \deg(p_{\mathcal{H}}, c) + N_0 \mathrm{mult}(p_{h_0,\mathcal{H}}, c) + \cdots + N_m \mathrm{mult}(p_{h_m,\mathcal{H}}, c) = 1. \end{aligned}$$

Therefore $l \in \mathbb{N}$. This, however contradicts Theorem 6.1. $\square$

9 Application

This section contains the main application of the results of the previous sections. In fact, we show that the Jacobian condition implies the invertibility when the topological degree of (f, g) is less than 6. Orevkov and Domrina proved this statement when the topological degree is less than 5 (cf. [O 2], [D-O], [D]). In this section we give a new proof for these known results about the topological degree and also extend it to the topological degree 6.

Proposition 9.1. *If $\Lambda(\mathcal{F}) \leq 6$, then all the possible multiplicity data for*

$\mathcal{F} \in T_{a,\mathrm{pole}}$ *is listed below:*

	type	$(D_{\mathcal{F}}, D_{g,\mathcal{F}})$	$(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}}))$	$\nu_{\mathcal{F}}$	$\Lambda(\mathcal{F})$
1	$(2,3)$	$(2,3)$	$(2,3)$	2	3
2	$(2,3)$	$(2,3)$	$(2,3)$	1	6
3	$(2,3)$	$(4,6)$	$(2,3)$	2	6
4	$(2,3)$	$(2,3)$	$(4,6)$	3	4
6	$(3,4)$	$(3,4)$	$(3,4)$	3	4
6	$(3,4)$	$(3,4)$	$(3,4)$	2	6
7	$(2,5)$	$(2,5)$	$(2,5)$	2	5
8	$(2,5)$	$(2,5)$	$(6,15)$	5	6
9	$(3,5)$	$(3,5)$	$(6,10)$	5	6
10	$(4,5)$	$(4,5)$	$(4,5)$	4	5
11	$(5,6)$	$(5,6)$	$(5,6)$	5	6

(23)

Proof.

By Proposition 5.7 we obtain that the type of the multiplicity data is (α, β) where $1 < \alpha < \beta \leq 6$ and $\gcd.(\alpha, \beta) = 1$.

First consider the type $(\alpha, \beta) = (2,3)$. By (ii) of Statement 5.2, in this case $\nu_{\mathcal{F}} \leq 3$. Therefore, by (19) one has $D_{g,\mathcal{F}} \cdot \deg(p_{\mathcal{F}}) \leq 18$. By applying (i) of Statement 5.2, we have the following possibilities:

(i) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (2,3)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (2,3)$;

(ii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (4,6)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (2,3)$;

(iii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (6,9)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (2,3)$;

(iv) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (2,3)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (4,6)$;

(v) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (2,3)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (6,9)$.

First consider (i). By Statement 5.2 one has $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. These two cases give (1) and (2) in the table.

Cosider (ii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 1$ or $\nu_{\mathcal{F}} = 2$. By (19) one has $\nu_{\mathcal{F}} \neq 1$. Therefore, the only remaining case is provided by (3) in the table.

Consider (iii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 2$, so we don't obtain new case.

Consider (iv). By Statement 5.2 one has $\nu_{\mathcal{F}} = 3$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} \neq 1$. The remaining $\nu_{\mathcal{F}} = 3$ case gives (4).

Consider (v). By Statement 5.2 one has $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 2$, so we don't obtain new case.

Consider the type $(\alpha, \beta) = (3, 4)$. By (ii) of Statement 5.2, in this case $\nu_{\mathcal{F}} \leq 4$. Therefore, by (19) one has $D_{g,\mathcal{F}} \cdot \deg(p_{\mathcal{F}}) \leq 24$.

By applying (i) of Statement 5.2, we have the following possibilities:

(i) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (3, 4)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (3, 4)$;

(ii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (6, 8)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (3, 4)$;

(iii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (3, 4)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (6, 8)$.

Consider (i). By Statement 5.2 one has $\nu_{\mathcal{F}} = 3$ or $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} \neq 1$. The remaining two cases give (5) and (6).

Consider (ii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 3$ or $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 3$, so we don't obtain new case.

Consider (iii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} \neq 1$, so we don't obtain new case.

Consider the type $(\alpha, \beta) = (2, 5)$. By (ii) of Statement 5.2, in this case $\nu_{\mathcal{F}} \leq 5$. Therefore, by (19) one has $D_{g,\mathcal{F}} \cdot \deg(p_{\mathcal{F}}) \leq 30$.

By applying (i) of Statement 5.2, we have the following possibilities:

(i) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (2, 5)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (2, 5)$;

(ii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (4, 10)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (2, 5)$;

(iii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (6, 15)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (2, 5)$;

(iv) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (2, 5)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (4, 10)$;

(v) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (2, 5)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (6, 15)$.

Consider (i). By Statement 5.2 one has $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} \neq 1$. The remaining $\nu_{\mathcal{F}} = 2$ case gives (7).

Consider (ii) and (iii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 2$, so we don't obtain new case.

Consider (iv). By Statement 5.2 one has $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} \neq 1$, so we don't obtain new case.

Consider (v). By Statement 5.2 one has $\nu_{\mathcal{F}} = 5$ or $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 2$. The remaining $\nu_{\mathcal{F}} = 5$ case gives (8).

Consider the type $(\alpha, \beta) = (3, 5)$. By (ii) of Statement 5.2, in this case $\nu_{\mathcal{F}} \leq 5$. Therefore, by (19) one has $D_{g,\mathcal{F}} \cdot \deg(p_{\mathcal{F}}) \leq 30$.

By applying (i) of Statement 5.2, we have the following possibilities:

(i) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (3, 5)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (3, 5)$;

(ii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (6, 10)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (3, 5)$;

(iii) $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (3,5)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (6,10)$.

Consider (i) and (ii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} \neq 1$, so we don't obtain new case. Consider (iii). By Statement 5.2 one has $\nu_{\mathcal{F}} = 5$ or $\nu_{\mathcal{F}} = 3$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 3$. The remaining $\nu_{\mathcal{F}} = 5$ case gives (9).

Consider the type $(\alpha, \beta) = (4,5)$. By (ii) of Statement 5.2, in this case $\nu_{\mathcal{F}} \leq 5$. Therefore, by (19) one has $D_{g,\mathcal{F}} \cdot \deg(p_{\mathcal{F}}) \leq 30$.

By applying (i) of Statement 5.2, we have the only possibility: $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (4,5)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (4,5)$. By Statement 5.2 one has $\nu_{\mathcal{F}} = 4$ or $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 2$, and the only remaining case $\nu_{\mathcal{F}} = 4$ gives (10).

Consider the type $(\alpha, \beta) = (5,6)$. By (ii) of Statement 5.2, in this case $\nu_{\mathcal{F}} \leq 6$. Therefore, by (19) one has $D_{g,\mathcal{F}} \cdot \deg(p_{\mathcal{F}}) \leq 36$.

Therefore, by applying (i) of Statement 5.2, we have only one possibility: $(D_{\mathcal{F}}, D_{g,\mathcal{F}}) = (5,6)$ and $(\deg(p_{\mathcal{F}}), \deg(p_{g,\mathcal{F}})) = (5,6)$. By Statement 5.2 one has $\nu_{\mathcal{F}} = 5$ or $\nu_{\mathcal{F}} = 2$ or $\nu_{\mathcal{F}} = 1$. By (19) one has $\nu_{\mathcal{F}} > 2$. The remaining $\nu_{\mathcal{F}} = 5$ case gives (11). Since there are no remaining types, our proof is finished. □

Notation 9.1. *Set $\mathcal{F} \in V_a \cup T_a^+$. Define*

$$Q(\mathcal{F}) := (D_{\mathcal{F}}, \deg(p_{\mathcal{F}}), \nu_{\mathcal{F}}, M_{\mathcal{F}}, \kappa_{\mathcal{F}}(1 - \pi(\mathcal{F}))).$$

Statement 9.1. *Assume $\mathcal{F} \in T_{a,\mathrm{pole}}$. Then $D_{\mathcal{F}} + D_{g,\mathcal{F}} = \kappa_{\mathcal{F}}(1 - \pi(\mathcal{F})))$.*

Proof. From Proposition 4.1 one has $d_{\mathcal{F}} + d_{g,\mathcal{F}} = 1 - u$. Therefore, $D_{\mathcal{F}} + D_{g,\mathcal{F}} = \kappa_{\mathcal{F}}(1 - u)$. □

Notation 9.2. *Set $\mathcal{F} \in V_a \cap T_a^{\searrow} \setminus \{(0,y)\}$ and $\mathcal{G} := \mathcal{F}^\circ$. We say that $\mathcal{G}$ is regular over $\mathcal{F}$ if for any $\mathcal{H} \in V_a \cap T_a^{\searrow} \setminus \{(0,y)\}$ with $\mathcal{H}^\circ = \mathcal{G}$ one has $\mathcal{H} = \mathcal{F}$.*

Statement 9.2. *Let $\mathcal{F} = (0,y)$. Then*

(i) $D_{\mathcal{F}} = d_{\mathcal{F}}$;

(ii) $\nu_{\mathcal{F}} = 1$;

(iii) $\kappa_{\mathcal{F}}(1 - \pi(\mathcal{F})) = 1$.

Statement 9.3. *Set* $\mathcal{F} \in T_a^{\searrow} \cap V_a$. *Assume* $\mathcal{G} := \mathcal{F} + c^* \in T_a^{\nearrow}$. *Set* $\mathcal{G} := I_P(v)$ *for some* $P \in \overline{R}_a \setminus R_a$. *Set* $\mathcal{H} = I_P(w)$ *such that* $\mathcal{H} \in T_{a,\mathrm{cv}}$. *Then*

$$\kappa_{\mathcal{H}}(\pi(\mathcal{H}) - 1) \geq \begin{cases} \frac{D_{\mathcal{F}}}{\mathrm{mult}(p_{\mathcal{F}}, c^*)} - \kappa_{\mathcal{F}}(\pi(\mathcal{F}) - 1) & \text{if } c^* \neq 0 \\ \frac{D_{\mathcal{F}}}{\nu_{\mathcal{F}} \mathrm{mult}(p_{\mathcal{F}}, c^*)} - \frac{\kappa_{\mathcal{F}}(\pi(\mathcal{F})-1)}{\nu_{\mathcal{F}}} & \text{if } c^* = 0. \end{cases} \tag{24}$$

Proof. Assume $\mathcal{F} = I_P(u)$. Then, by Statement 3.11 one has $0 = d_{\mathcal{H}} \geq d_{\mathcal{F}} - (v-u)\mathrm{mult}(p_{\mathcal{F}}, c^*)$. Therefore $v - u \geq \frac{d_{\mathcal{F}}}{\mathrm{mult}(p_{\mathcal{F}}, c^*)}$. Moreover, if $c \neq 0$, then evidently $\kappa_{\mathcal{H}} \geq \kappa_{\mathcal{F}}$. Therefore

$$\kappa_{\mathcal{H}}(w-1) \geq \kappa_{\mathcal{F}}(w-1) = \kappa_{\mathcal{F}}(w-u) - \kappa_{\mathcal{F}}(u-1) \geq$$
$$\kappa_{\mathcal{F}} \frac{d_{\mathcal{F}}}{\mathrm{mult}(p_{\mathcal{F}}, c^*)} - \kappa_{\mathcal{F}}(u-1) = \geq \frac{D_{\mathcal{F}}}{\mathrm{mult}(p_{\mathcal{F}}, c_1)} - \kappa_{\mathcal{H}}(\pi(\mathcal{F}) - 1).$$

This proves our statement in the case $c^* \neq 0$. In the case $c^* = 0$ the same argument works, using the trivial inequality $\kappa_{\mathcal{H}} \geq \frac{\kappa_{\mathcal{F}}}{\nu_{\mathcal{F}}}$. □

Notation 9.3. *Set* $\mathcal{F} \in V_a \cap T_a^{\searrow}$. *Define*

$$Y(\mathcal{F}) := \{\mathcal{H} :\in T_{a,\mathrm{cv}} : \ \exists P \in \overline{R}_a \setminus R_a : \ \mathcal{F} = I_P(u) \ \text{ and } \ \mathcal{H} = I_P(\pi(\mathcal{H}))\}.$$

Define

$$\lambda_{\mathcal{F}} := \sum_{\mathcal{H} \in Y(\mathcal{F})} \kappa_{\mathcal{H}}(\pi(\mathcal{H}) - 1).$$

Statement 9.4. *Let* $\mathcal{F}_1, \ldots, \mathcal{F}_n \in V_a \cap T_a^{\searrow}$ *be pairwise different. Assume that for* $\psi \in \mathbb{N}$ *one has* $\psi l_f < k_f$. *Then*

$$\sum_{i=1}^{n} \lambda_{\mathcal{F}_i} \leq td(f,g) - 1 - \psi. \tag{25}$$

In particular,

$$\sum_{i=1}^{n} \lambda_{\mathcal{F}_i} \leq td(f,g) - 2. \tag{26}$$

Proof. Since $(0,x) \in T_a^{\nearrow}$, by Statement 7.3, there exists $\mathcal{G} \in T_{a,\mathrm{cv}} \cap T_{a,x}$. From Statement 3.11, one has $\pi(\mathcal{G}) > \psi$. Therefore, $\kappa_{\mathcal{G}}(\pi(\mathcal{G}) - 1) > \psi - 1$. Since $\kappa_{\mathcal{G}}(\pi(\mathcal{G}) - 1) \in \mathbb{N}$, one has $\kappa_{\mathcal{G}}(\pi(\mathcal{G}) - 1) \geq \psi$.

Therefore, from Corollary 7.1 one has

$$\sum_{i=1}^{n} \lambda_{\mathcal{F}_i} + \psi \leq td(f,g) - 1,$$

which is equivalent with our statement. □

Proposition 9.2. *Set $\mathcal{F} \in T_{a,\text{pole}}$. With recursion define the following finite sequence, $\{\mathcal{F}_i\}_{i=0}^n$: Start with $\mathcal{F}_0 := \mathcal{F}$. Assume that $\mathcal{F}_i$ is already defined. There are two cases.*

(i) $\mathcal{F}_i \neq (0, y)$. Then set $\mathcal{F}_{i+1} := \mathcal{F}^\circ$.

(ii) $\mathcal{F}_i = (0, y)$. Then we stop.

We call the sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n = (0, y)$ the characteristic sequence of $\mathcal{F}$.

Statement 9.5. *Set $\mathcal{F} \in T_{a,pole}$. Let $\mathcal{F}_0, \ldots, \mathcal{F}_n$ be the characteristic sequence of $\mathcal{F}$. Then*

$$\sum_{i=1}^{n} \lambda_{\mathcal{F}_i} \leq td(f, g) - 2.$$

Proof. We apply Statement 9.4 for $\mathcal{F}_0, \ldots, \mathcal{F}_n$. $\square$

Proposition 9.3. *Let $\mathcal{F}, \mathcal{G} \in V_a \cap T_a^{\searrow}$ such that $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. Set $\pi(\mathcal{F}) = u$, $\pi(\mathcal{G}) = v$. Then there exists $P \in \overline{R}_a \setminus R_a$ such that the Puiseux characteristics of P is $(\kappa, \beta_1, \ldots, \beta_s)$ and $v = \alpha_j := \frac{\beta_j}{\kappa}$. Moreover, exactly one of the following four possibilities must hold:*

(I) $\mathcal{F} \in V_{2,a} \setminus V_{1,a}$;

(II) $\mathcal{F} \in V_{1,a}$ and $u = \alpha_{j-1} := \frac{\beta_{j-1}}{\kappa}$;

(III) $\mathcal{F} \in V_{1,a}$ and $u > \alpha_{j-1} := \frac{\beta_{j-1}}{\kappa}$;

(IV) $\mathcal{F} = (0, y)$ and $\mathcal{F} \notin V_{1,a} \cup V_{2,a}$.

In particular, via Proposition 8.1, in the cases (I) and (II), there exists $n \in \mathbb{N}^*$ such that:

(a) $u = v - \frac{n}{\kappa_{\mathcal{G}}}$;

(b) $\frac{\deg(p)}{\deg q} = \frac{D_{\mathcal{G}} + n \deg(p_{\mathcal{G}})}{i(\kappa_{\mathcal{G}}(1-v)+n)}$;

(c) $D_{\mathcal{F}} = \frac{D_{\mathcal{G}} + n \deg(p_{\mathcal{G}})}{\nu_{\mathcal{G}}}$;

(d) $\kappa_{\mathcal{F}}(1 - \pi(\mathcal{F})) = \frac{(1-\pi(\mathcal{G}))\kappa_{\mathcal{G}}+n}{\nu_{\mathcal{G}}}$.

Assume (III) holds. Let $\nu := \nu_{\mathcal{F}}$. Then there exists $n \in \mathbb{N}^*$ such that

(e) $u = v - \frac{n}{\nu \kappa_{\mathcal{G}}}$;

(f) $\frac{\deg(p)}{\deg(q)} = \frac{\nu D_{\mathcal{G}}+n\deg(p_{\mathcal{G}})}{i(\nu\kappa_{\mathcal{G}}(1-v)+n)}$;

(g) $D_{\mathcal{F}} = \frac{\nu D_{\mathcal{G}}+n\deg(p_{\mathcal{G}})}{\nu_{\mathcal{F}}}$;

(h) $\kappa_{\mathcal{F}}(1-\pi(\mathcal{F})) = \frac{\nu(1-\pi(\mathcal{G}))\kappa_{\mathcal{G}}+n}{\nu_{\mathcal{F}}}$.

Finally, if (IV) holds, then

(i) $\nu_{\mathcal{G}} = \kappa_{\mathcal{G}}$;

(j) $(1-v)\kappa_{\mathcal{G}} < \nu_{\mathcal{G}}$;

(k) $d_{\mathcal{F}} = \frac{D_{\mathcal{G}}+(\nu-(1-v)\kappa_{\mathcal{G}})\deg(p_{\mathcal{G}})}{\nu_{\mathcal{G}}} \in \mathbb{N}$;

(l) $d_{\mathcal{F}} < \deg(p_{\mathcal{G}})$;

(m) $d_{\mathcal{F}}\frac{M_{\mathcal{G}}}{\deg(p_{\mathcal{G}})} \in \mathbb{N}$.

Proof. Property (a) is a consequence of the definition of V_a. From (a) and Statement 3.17 one has

$$D_{\mathcal{F}} = \kappa_{\mathcal{F}}d_{\mathcal{F}} = \frac{\kappa_{\mathcal{G}}}{\nu_{\mathcal{G}}}(d_{\mathcal{G}}+(v-u)\deg(p_{\mathcal{G}})) = \frac{D_{\mathcal{G}}+n\deg(p_{\mathcal{G}})}{\nu_{\mathcal{G}}},$$

which gives (c).

From (a) one has

$$\kappa_{\mathcal{F}}(1-\pi(\mathcal{F})) = \frac{\kappa_{\mathcal{G}}}{\nu_{\mathcal{G}}}(1-u) = \frac{\kappa_{\mathcal{G}}}{\nu_{\mathcal{G}}}((1-v)+v-u) = \frac{(1-\pi(\mathcal{G}))\kappa_{\mathcal{G}}+n}{\nu_{\mathcal{G}}},$$

which proves (d). Moreover, (a), (c), (d) and Statement 8.2 imply (b). The remaining Statements can be proved the same way as above.

Statement 9.6. *Set $\mathcal{G} \in T_a^{\searrow}$. Assume that $\mathcal{G}^\circ := \mathcal{F}$ is regular over $\mathcal{G}$ and $Q(\mathcal{G}) = (j,2j,3,2,5)$ for some $j \in \mathbb{N}^*$. Then we have the following possibilities:*

(i) $M_{\mathcal{F}} = 1$;

(ii) $\lambda_{\mathcal{F}} \geq 3$;

(iii) $Q(\mathcal{F}) = (7j,21j,7,3,5)$ *and* $\lambda_{\mathcal{F}} \geq 2$;

(iv) $Q(\mathcal{F}) = (5j,20j,5,4,4)$ *and* $\lambda_{\mathcal{F}} \geq 2$;

(v) $Q(\mathcal{F}) = ((6s+3)j,(4s+2)j,2s+1,2,3s+3)$ *and* $\lambda_{\mathcal{F}} = 0$.

Proof. Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. We use the notations of Propositions 8.1 and 9.3. Since (j) does not hold, the case (IV) of Proposition 9.3 is impossible. Therefore, by Statement 3.16, p has more than one root.

By Statement 8.4 one has $\mathrm{mult}(p, c) \mid 2$. From Statement 6.2 and from the regularity assumption we obtain that for any $c^* \in \mathbb{C} \setminus \{c\}$ such that $\mathcal{F} * c$ exists, one has $\mathrm{mult}(p, c^*) < \mathrm{mult}(p, c)$. Assume first that $\mathrm{mult}(p, c) = 1$. Then by Statements 3.16 and 3.18 one has $p(\eta) = \ominus(\eta^\nu - c^\nu)$ for some $\nu = \nu_{\mathcal{F}}$. Therefore $\deg(q) = n\nu + 1$ for some $n \in \mathbb{N}^*$. From (v) we obtain that $M_{\mathcal{F}} = 1$.

Assume $\mathrm{mult}(p, c) = 2$. Then $i = \frac{\deg(p_{\mathcal{G}})}{M_{\mathcal{G}}} = j$. Consider the case (I) of Proposition 9.3. Then $p(\eta) = \ominus(\eta - c)^2(\eta - c_1) \cdot \dots \cdot (\eta - c_k)$, and $q(\eta) = \ominus(\eta - c)(\eta - c_1) \cdot \dots \cdot (\eta - c_{k+l})$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_{k+l} \in \mathbb{C}$ are pairwise different. From Statement 3.16 one has $k > 0$. From Statement 8.2 one has $l = 0$. Therefore, by (v) of Proposition 8.1 we obtain $M_{\mathcal{F}} = \gcd.(k + 2, k + 1) = 1$.

Consider (II). In this case $\nu = \nu_{\mathcal{F}} > 1$. Moreover, by Statement 3.18 we obtain that there are two possibilities:

(a) $p(\eta) = \ominus(\eta^\nu - c^\nu)^2(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_k^\nu)$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)^2(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_{k+l}^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_{k+l} \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_{k+l}^\nu$ are pairwise different.

(b) $p(\eta) = \ominus\eta(\eta^\nu - c^\nu)^2(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_k^\nu)$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)^2(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_{k+l}^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_{k+l} \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_{k+l}^\nu$ are pairwise different.

Consider (a). Assume $k \neq 0$. Then by Statement 8.2, $l = 0$. From (b) of Proposition 9.3 we obtain

$$\frac{(k+2)\nu}{(k+1)\nu + 1} = \frac{\deg(p)}{\deg(q)} = \frac{D_{\mathcal{G}} + n\deg(p_{\mathcal{G}})}{i(\kappa_{\mathcal{G}}(1 - v) + n} = \frac{1 + 2n}{5 + n}.$$

From (d) of Proposition9.3 we obtain $n = 3m + 1$ for some $m \in \mathbb{N}$. By the former conditions, we are looking for the solutuions of the above Diophantic equation with variables $k, n \in \mathbb{N}^*$ and $\nu \in \mathbb{N}^* \setminus \{1\}$ and additional relation $n = 3m + 1$:

(A) $k = 1$, $n = 10$ and $\nu = 7$. Therefore $\deg(p) = 21$ and $\deg(q) = 15$;

(B) $k = 1$, $n = 13$ and $\nu = 25$. Therefore, $\deg(p) = 75$ and $\deg(q) = 51$;

(C) $k = 2$, $n = 7$ and $\nu = 5$. Therefore, $\deg(p) = 20$ and $\deg(q) = 16$.

Consider (A). We have

$$D_{\mathcal{F}} = \kappa_{\mathcal{F}} d_{\mathcal{F}} = \frac{\kappa_{\mathcal{G}}}{\nu_{\mathcal{G}}}(d_{\mathcal{G}} + (v - u)\deg(p_{\mathcal{G}})) = \frac{D_{\mathcal{G}} + n\deg(p_{\mathcal{G}})}{\nu_{\mathcal{F}}} = 7j.$$

$\deg(p_{\mathcal{F}}) = j\deg(p) = 21j$.
$M_{\mathcal{F}} = \gcd.(\deg(p), \deg(q)) = 3$.

$$\kappa_{\mathcal{F}}(1-\pi(\mathcal{F})) = \frac{\kappa_{\mathcal{G}}}{\nu_{\mathcal{G}}}(1-u) = \frac{\kappa_{\mathcal{G}}}{\nu_{\mathcal{G}}}((1-v)+v-u) = \frac{(1-\pi(\mathcal{G}))\kappa_{\mathcal{G}}+n}{\nu_{\mathcal{F}}} = 5.$$

Set $\mathcal{G}^* := \mathcal{F} + c_1$. Assume $\mathcal{G}^* = I_Q(v^*)$ for some $Q \in \overline{R}_a \setminus R_a$ and $v^* \in \mathbb{Q}$. Since $\mathcal{G}^* \in T_a^{\nearrow}$, there exists $\mathcal{H} = I_Q(w) \in T_{a,\mathrm{cv}}$. Moreover, from Statement 9.3 we obtain $\lambda_{\mathcal{F}} \geq 2$. This gives (iii).

Consider (B). We have $D_{\mathcal{F}} = 9j$, $\deg(p_{\mathcal{F}}) = 75j$, $M_{\mathcal{F}} = 3$ and $\kappa_{\mathcal{F}}(1 - \pi(\mathcal{F})) = 6$. From Statement 9.3 we obtain $\lambda_{\mathcal{F}} \geq 3$.

Consider (C). We have $D_{\mathcal{F}} = 5j$ $\deg(p_{\mathcal{F}}) = 20j$, $M_{\mathcal{F}} = 4$, $\kappa_{\mathcal{F}}(1-\pi(\mathcal{F})) = 4$. Since $k = 2$, $\lambda_{\mathcal{F}} \geq 2$. This gives (iv)

Consider the case $k = 0$. Then $\deg(p) = 2\nu$, $\deg(q) = (l+1)\nu + 1$. Therefore, by (v) of Proposition 8.1 one has $M_{\mathcal{F}} = \gcd.(\deg(p), \deg(q))$ is 1 or 2. Moreover, $M_{\mathcal{F}} = 2$ if and only if l is even and ν is odd. From (b) of Proposition 9.3 one has

$$\frac{2\nu}{(l+1)\nu+1} = \frac{2n+1}{n+5}.$$

The solutions of this Diophantic equation with the assumptions above takes the form: $l = 0$, $n = 9s + 4$, and $\nu = 2s + 1$ for some $s \in \mathbb{N}^*$. Consider (A). From (c) of Proposition 9.3 one has $D_{\mathcal{F}} = (6s + 3)j$.
$\deg(p_{\mathcal{F}}) = i\deg(p) = (4s + 2)j$.
From (v) of Proposition 8.1 one has $M_{\mathcal{F}} = 2$.
From (d) of Proposition 9.3 one has $\kappa_{\mathcal{F}}(1 - (\pi(\mathcal{F})) = 3s + 3$. Since the only c^* for which $\mathcal{F} + c^*$ exists, is c one gets $\lambda_{\mathcal{F}} = 0$. This gives (v).

Consider (b). Then by Statement 8.2 one has $l = 0$. In this case, by (v) of Proposition 8.1 one has $M_{\mathcal{F}} = \gcd.(\deg(p), \deg(q)) = \gcd.((k+2)\nu + 1, (k+1)\nu + 1) = 1$.

Consider the type (III) of Proposition 9.3. Then by Statements 3.16 and 8.2 one has $p(\eta) = \eta^2(\eta^\nu - c_1^\nu)\dots(\eta^\nu - c_k^\nu)$ and $q(\eta) = \eta(\eta^\nu - c_1^\nu)\dots(\eta^\nu - c_k^\nu)$ In this case, by (v) of Proposition 8.1 one has $M_{\mathcal{F}} = 1$.

Since there are no remaining cases, our proof is finished. $\square$.

Statement 9.7. *Set $\mathcal{G} \in T_a^{\searrow}$. Assume that $\mathcal{F} := \mathcal{G}^\circ$ is regular over $\mathcal{F}$. Assume that $Q(\mathcal{G}) = (j, 3j, 7, 3, 5)$. Then we have the following possibilities:*

(i) $M_{\mathcal{G}} = 1$;

(ii) $\lambda_{\mathcal{F}} \geq 1$;

(iii) $\mathcal{F} = (0, y)$ and $Q(\mathcal{F}) = ((j, 3j, 1, M, 1)$ for some $M \in \mathbb{N}$;

(iv) $Q(\mathcal{F}) = ((6s+4)j, (9s+6)j, 3s+2, 3, 2s+2)$.

Proof. Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. We use the notations of Propositions 8.1 and 9.3. Consider first the case (IV) of Proposition 9.3. Then from (k) of Proposition 9.3 one has $d_{\mathcal{F}} = D_{\mathcal{F}} = 7j$. Moreover, $\deg(p_{\mathcal{F}}) = \deg(p_{\mathcal{G}}) = 21j$. This gives (iii).

In any other cases, by Statement 3.16 one has that p has more than one root. By Statement 8.4 one has that $\mathrm{mult}(p, c) \mid 3$. From Statement 6.2 and from the regularity assumption we obtain that for any $c^* \in \mathbb{C} \setminus \{c\}$ such that $\mathcal{F} * c$ exists, one has $\mathrm{mult}(p, c^*) < \mathrm{mult}(p, c)$. Assume first $\mathrm{mult}(p, c) = 1$. Then by Statements 3.16 and 3.18 one has $p(\eta) = \ominus(\eta^\nu - c^\nu)$ for some $\nu = \nu_{\mathcal{F}}$. Therefore $\deg(q) = n\nu + 1$ for some $n \in \mathbb{N}^*$. From (v) we obtain that $M_{\mathcal{F}} = 1$.

Assume $\mathrm{mult}(p, c) = 3$. Then $i = j$.

Consider the case (I) of Proposition 9.3. Then $p(\eta) = \ominus(\eta - c)^3(\eta - c_1)\dots(\eta - c_k)$ for some $k \in \mathbb{N}^*$, and $c_1, \dots, c_k \in \mathbb{C}\{c\}$ pairwise different points. In this case $\mathcal{F} + c_1 \in T_a^{\nearrow}$. Therefore $\lambda_{\mathcal{F}} \geq 1$.

Consider the case (II) of Proposition 9.3. In this case $\nu := \nu_{\mathcal{F}} > 1$. Moreover, by Statement 3.18 we obtain that there are two possibilities:

(a) $p(\eta) = \ominus(\eta^\nu - c^\nu)^3(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_k^\nu)$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_{k+l}^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_{k+l} \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_{k+l}^\nu$ are pairwise different.

(b) $p(\eta) = \ominus\eta(\eta^\nu - c^\nu)^3(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_k^\nu)$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_{k+l}^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_{k+l} \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_{k+l}^\nu$ are pairwise different.

Consider (a). Then in the case $k > 0$ one has $\lambda_{\mathcal{F}} \geq 1$. Assume $k = 0$. Then (b) of Proposition 9.3 gives

$$\frac{3\nu}{(l+1)\nu + 1} = \frac{1+3n}{5+n}.$$

From (d) of Proposition 9.3 one has $n = 7m + 2$ for some $m \in \mathbb{N}$. The above Diophantic equation has the following solution with the condition above: $l = 0$, $n = 14s + 9$ and $\nu = 3s + 2$ for some $s \in \mathbb{N}$. Therefore $\deg(p) = 9s + 6$, $\deg(q) = 3s + 3$. In this case (c) of Proposition 9.3 gives $D_{\mathcal{F}} = (6s+4)j$. Moreover, $\deg(p_{\mathcal{F}}) = (9s+6)j$ and from (v) of Proposition 8.1 one has $M_{\mathcal{F}} = 3$. By (d) of Proposition 9.3 one has $\kappa_{\mathcal{F}}(1 - \pi(\mathcal{F})) = 2s + 2$. This gives (iv).

In the case (b) $\mathcal{F} + 0 \in T_a^{\nearrow}$, therefore $\lambda_{\mathcal{F}} \geq 1$.

In the case (III) of Proposition 9.3 there exists $c^* \in \mathbb{C}^*$ such that $\mathcal{H} := \mathcal{F} + c^*$ exists. By the regularity condition, $\mathcal{H} \in T_a^{\nearrow}$. Therefore $\lambda_{\mathcal{F}} \geq 1$. Since there are no remaining cases, our proof is finished. $\square$

Statement 9.8. *Set $\mathcal{G} \in T_a^{\searrow}$. Assume that $\mathcal{F} := \mathcal{G}^\circ$ is regular over $\mathcal{F}$. Assume that $Q(\mathcal{G}) = (j, 4j, 5, 4, 4)$. Then we have the following possibilities:*

(i) $M_\mathcal{G} = 1$;

(ii) $\lambda_\mathcal{F} \geq 1$;

(iii) $\mathcal{F} = (0, y)$ and $Q(\mathcal{F}) = ((j, 4j, 1, M, 1)$ for some $M \in \mathbb{N}$;

(iv) $Q(\mathcal{G}) = ((12s + 9)j, (16s + 12)j, 4s + 3, 4, 3s + 3)$.

Proof. Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. We use the notations of Propositions 8.1 and 9.3. Consider first the case (IV) of Proposition 9.3. Then from (k) of Proposition 9.3 one has $d_\mathcal{F} = D_\mathcal{F} = j$. Moreover, $\deg(p_\mathcal{F}) = \deg(p_\mathcal{G}) = 4j$. This gives (iii).

In any other cases, by Statement 3.16 one has that p has more than one root.

The same way as in Statement 9.7 can be seen that both the cases (I) and (III) of Proposition 9.3 imply $\lambda_\mathcal{F} \geq 1$.

Consider the case (II) of Proposition 9.3. In this case $\nu := \nu_\mathcal{F} > 1$. By Statement 8.4 one has that $\mathrm{mult}(p, c) \mid 4$. From Statement 6.2 and from the regularity assumption we obtain that for any $c^* \in \mathbb{C} \setminus \{c\}$ such that $\mathcal{F} * c$ exists, one has $\mathrm{mult}(p, c^*) < \mathrm{mult}(p, c)$. Assume first $\mathrm{mult}(p, c) = 1$. Then by Statements 3.16 and 3.18 one has $p(\eta) = \ominus(\eta^\nu - c^\nu)$ for some $\nu = \nu_\mathcal{F}$. Therefore $\deg(q) = n\nu + 1$ for some $n \in \mathbb{N}^*$. From (v) we obtain that $M_\mathcal{F} = 1$. Assume $\mathrm{mult}(p, c) = 2$. Then $i = 2j$.

In the case $\lambda_\mathcal{F} = 0$, by Statement 3.18 we obtain $p(\eta) = \ominus(\eta^\nu - c^\nu)^2$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_l^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_l \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_l^\nu$ are pairwise different. In the case $l = 0$ from (v) of Proposition 8.1 we obtain $M_\mathcal{F} = 1$. From (b) of Proposition 9.3 one has

$$\frac{2\nu}{(l+1)\nu + 1} = \frac{1 + 4n}{8 + 2n}.$$

From (d) of Proposition 9.3 one has $n = 5m + 1$ for some $m \in \mathbb{N}$. The above Diophantic equation has no solution with the conditions $n = 5m + 1$ and $l > 0$.

Consider the case $\mathrm{mult}(p, c) = 4$. In the case $\lambda_\mathcal{F} = 0$, by Statement 3.18 we obtain $p(\eta) = \ominus(\eta^\nu - c^\nu)^4$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_l^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_l \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_l^\nu$ are pairwise different.

In the case $l = 0$ from (v) of Proposition 8.1 we obtain $M_\mathcal{F} = 1$. From (b) of Proposition 9.3 one has

$$\frac{2\nu}{(l+1)\nu + 1} = \frac{1 + 4n}{8 + 2n}.$$

From (d) of Proposition 9.3 one has $n = 5m + 1$. The above Diophantic equation has the following solution: $l = 0$, $n = 15s + 11$ and $\nu = 4s + 3$ for some $s \in \mathbb{N}$. Therefore $\deg(p) = 16s + 12$, $\deg(q) = 4s + 4$. In this case (c) of Proposition 9.3 gives $D_{\mathcal{F}} = (12s + 9)j$ Moreover, $\deg(p_{\mathcal{F}}) = (16s + 12)j$ and from (v) of Proposition 8.1 one has $M_{\mathcal{F}} = 4$. By (d) of Proposition 9.3 one has $\kappa_{\mathcal{F}}(1 - \pi(\mathcal{F})) = 3s + 3$. This gives (iv).

Since there are no remaining cases, our proof is finished. $\square$

Statement 9.9. *Set $\mathcal{G} \in T_a^{\searrow}$. Assume that $\mathcal{F} := \mathcal{G}^\circ$ is regular over $\mathcal{F}$. Assume that $Q(\mathcal{G}) = ((12s+9)j, (16s+12)j, 4s+3, 4, 3s+3)$. Then we have the following possibilities:*

(i) $M_{\mathcal{G}} = 1$;

(ii) $\lambda_{\mathcal{F}} \geq 1$;

(iii) $\mathcal{F} = (0, y)$ and $Q(\mathcal{F}) = ((4s+3)j, (16s+12)j, 1, M, 1)$ for some $M \in \mathbb{N}$.

Proof. Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. We use the notations of Propositions 8.1 and 9.3. Consider first the case (IV) of Proposition 9.3. Then from (k) of Proposition 9.3 one has $d_{\mathcal{F}} = D_{\mathcal{F}} = (4s + 3)j$. Moreover, $\deg(p_{\mathcal{F}}) = \deg(p_{\mathcal{G}}) = (16s + 12)j$. This gives (iii).

In any other cases, by Statement 3.16 one has that p has more than one root.

The same way as in Statement 9.7 can be seen that both the cases (I) and (III) of Proposition 9.3 imply $\lambda_{\mathcal{F}} \geq 1$.

Consider the case (II) of Proposition 9.3. In this case $\nu := \nu_{\mathcal{F}} > 1$. By Statement 8.4 one has that $\mathrm{mult}(p, c) \mid 4$. From Statement 6.2 and from the regularity assumption we obtain that for any $c^* \in \mathbb{C} \setminus \{c\}$ such that $\mathcal{F} * c$ exists, one has $\mathrm{mult}(p, c^*) < \mathrm{mult}(p, c)$. Assume first $\mathrm{mult}(p, c) = 1$. Then by Statements 3.16 and 3.18 one has $p(\eta) = \ominus(\eta^\nu - c^\nu)$ for some $\nu = \nu_{\mathcal{F}}$. Therefore $\deg(q) = n\nu + 1$ for some $n \in \mathbb{N}^*$. From (v) we obtain that $M_{\mathcal{F}} = 1$. Assume $\mathrm{mult}(p, c) = 2$. Then $i = 2j(4s + 3)$.

In the case $\lambda_{\mathcal{F}} = 0$, by Statement 3.18 we obtain $p(\eta) = \ominus(\eta^\nu - c^\nu)^2$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \cdots \cdot (\eta^\nu - c_l^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \ldots, c_l \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \ldots, c_l^\nu$ are pairwise different. From (b) of Proposition 9.3 one has

$$\frac{2\nu}{(l+1)\nu + 1} = \frac{3 + 4n}{2(3s + 3 + n)} = \frac{3 + 4s + 4m(4s + 3)}{2(4s + 3 + (4s + 3)m)} = \frac{4m + 1}{2m + 2},$$

since from (d) of Proposition 9.3 one has $n = s + m(4s + 3)$ for some $m \in \mathbb{N}$. The above Diophantic equation has no solution.

Consider the case $\mathrm{mult}(p, c) = 4$. Then $i = j(4s+3)$. In the case $\lambda_{\mathcal{F}} = 0$, by Statement 3.18 we obtain $p(\eta) = \ominus(\eta^\nu - c^\nu)^4$, and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_l^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_l \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_l^\nu$ are pairwise different.

From (b) of Proposition 9.3 one has

$$\frac{4\nu}{(l+1)\nu+1} = \frac{3+4n}{3s+3+n} = \frac{3+4s+4m(4s+3)}{4s+3+(4s+3)m} = \frac{4m+1}{m+1},$$

since from (d) of Proposition 9.3 one has $n = s + m(4s+3)$ for some $m \in \mathbb{N}$. The above Diophantic equation has no solution.

Since there are no remaining cases, our proof is finished.

Statement 9.10. *Set $\mathcal{G} \in T_a^{\searrow}$. Assume that $\mathcal{F} := \mathcal{G}^\circ$ is regular over $\mathcal{F}$. Assume that $Q(\mathcal{G}) = ((6s+4)j, (9s+6)j, 3s+2, 3, 2s+2)$. Then we have the following possibilities:*

(i) $M_{\mathcal{G}} = 1$;

(ii) $\lambda_{\mathcal{F}} \geq 1$;

(iii) $\mathcal{F} = (0, y)$ and $Q(\mathcal{F}) = (((3s+2)j, (9s+6)j, 1, M, 1)$ for some $M \in \mathbb{N}$;

(iv) $Q(\mathcal{G}) = ((12s+9)j, (16s+12)j, 4s+3, 4, 3s+3)$.

Proof.

Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. We use the notations of Propositions 8.1 and 9.3. Consider first the case (IV) of Proposition 9.3. Then from (k) of Proposition 9.3 one has $d_{\mathcal{F}} = D_{\mathcal{F}} = (3s+2)j$. Moreover, $\deg(p_{\mathcal{F}}) = \deg(p_{\mathcal{G}}) = (9s+6)j$. This gives (iii).

In any other cases, by Statement 3.16 one has that p has more than one root.

The same way as in Statement 9.7 can be seen that both the cases (I) and (III) of Proposition 9.3 imply $\lambda_{\mathcal{F}} \geq 1$.

Consider the case (II) of Proposition 9.3. In this case $\nu := \nu_{\mathcal{F}} > 1$. By Statement 8.4 one has that $\mathrm{mult}(p, c) \mid 3$. From Statement 6.2 and from the regularity assumption we obtain that for any $c^* \in \mathbb{C} \setminus \{c\}$ such that $\mathcal{F} * c$ exists, one has $\mathrm{mult}(p, c^*) < \mathrm{mult}(p, c)$. Assume first $\mathrm{mult}(p, c) = 1$. Then by Statements 3.16 and 3.18 one has $p(\eta) = \ominus(\eta^\nu - c^\nu)$ for some $\nu = \nu_{\mathcal{F}}$. Therefore $\deg(q) = l\nu + 1$ for some $l \in \mathbb{N}^*$. From (v) we obtain that $M_{\mathcal{F}} = 1$. Assume $\mathrm{mult}(p, c) = 3$. Then $i = j(3s+2)$.

In the case $\lambda_{\mathcal{F}} = 0$, by Statement 3.18 we obtain $p(\eta) = \ominus(\eta^\nu - c^\nu)^3$, itttt and $q(\eta) = \ominus\eta(\eta^\nu - c^\nu)(\eta^\nu - c_1^\nu) \cdot \dots \cdot (\eta^\nu - c_l^\nu)$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_l \in \mathbb{C}^*$ and $c^\nu, c_1^\nu, \dots, c_l^\nu$ are pairwise different. From (b) of Proposition 9.3 one has

$$\frac{3\nu}{(l+1)\nu+1} = \frac{2+3n}{2s+2+n} = \frac{3s+2+3m(3s+2)}{3s+2+(3s+2)m} = \frac{3m+1}{m+1},$$

since from (d) of Proposition 9.3 one has $n = s + m(3s+2)$ for some $m \in \mathbb{N}$. The above Diophantic equation has no solution.

Since there are no remaining cases, our proof is finished. $\square$.

Statement 9.11. *Set $\mathcal{G} \in T_a^{\searrow}$. Assume that $\mathcal{G}^\circ := \mathcal{F}$ is regular over $\mathcal{G}$ and $Q(\mathcal{G}) = ((6s+3)j, (4s+2)j, 2s+1, 2, 3s+3)$ for some $j \in \mathbb{N}^*$. Then we have the following possibilities:*

(i) $M_{\mathcal{F}} = 1$;

(ii) $\lambda_{\mathcal{F}} \geq 3$;

(iii) $Q(\mathcal{F}) = (5(2s+1)j, 20(2s+1)j, 5, 4, 4)$ *and* $\lambda_{\mathcal{F}} \geq 2$;

(iv) $Q(\mathcal{F}) = (7(2s+1)j, 21(2s+1)j, 7, 3, 5)$ *and* $\lambda_{\mathcal{F}} \geq 2$;

(v) $Q(\mathcal{F}) = (3(2s+1)(2\phi+1)j, 2(2s+1)(2\phi+1)j, 2\phi+1, 2, 3\phi+3)$ *and* $\lambda_{\mathcal{F}} = 0$.

Proof. Assume $\mathcal{G} = \mathcal{F} + c$ for some $c \in \mathbb{C}$. We use the notations of Propositions 8.1 and 9.3. Since (j) does not hold, the case (IV) of Proposition 9.3 is impossible. Therefore, by Statement 3.16, p has more than one root.

By Statement 8.4 one has $\mathrm{mult}(p, c) \mid 2$. From Statement 6.2 and from the regularity assumption we obtain that for any $c^* \in \mathbb{C} \setminus \{c\}$ such that $\mathcal{F} * c$ exists, one has $\mathrm{mult}(p, c^*) < \mathrm{mult}(p, c)$. Assume first that $\mathrm{mult}(p, c) = 1$. Then by Statements 3.16 and 3.18 one has $p(\eta) = \ominus(\eta^\nu - c^\nu)$ for some $\nu = \nu_{\mathcal{F}}$. Therefore $\deg(q) = n\nu + 1$ for some $n \in \mathbb{N}^*$. From (v) we obtain that $M_{\mathcal{F}} = 1$.

Assume $\mathrm{mult}(p, c) = 2$. Then $i = \frac{\deg(p_{\mathcal{G}})}{M_{\mathcal{G}}} = (2s+1)j$. Consider the case (I) of Proposition 9.3. Then $p(\eta) = \ominus(\eta - c)^2(\eta - c_1) \cdot \dots \cdot (\eta - c_k)$, and $q(\eta) = \ominus(\eta - c)(\eta - c_1) \cdot \dots \cdot (\eta - c_{k+l})$ where $l \in \mathbb{N}$, $c, c_1, \dots, c_{k+l} \in \mathbb{C}$ are pairwise different. From Statement 3.16 one has $k > 0$. From Statement 8.2 one has $l = 0$. Therefore, by (v) of Proposition 8.1 we obtain $M_{\mathcal{F}} = \gcd.(k+2, k+1) = 1$.

Consider (II). In this case $\nu = \nu_{\mathcal{F}} > 1$. Moreover, by Statement 3.18 we obtain that there are two possibilities:

(a) $p(\eta)=\ominus(\eta^\nu-c^\nu)^2(\eta^\nu-c_1^\nu)\cdot\dots\cdot(\eta^\nu-c_k^\nu)$, and $q(\eta)=\ominus\eta(\eta^\nu-c^\nu)^2(\eta^\nu-c_1^\nu)\cdot\dots\cdot(\eta^\nu-c_{k+l}^\nu)$ where $l\in\mathbb{N}$, $c,c_1,\dots,c_{k+l}\in\mathbb{C}^*$ and $c^\nu,c_1^\nu,\dots,c_{k+l}^\nu$ are pairwise different.

(b) $p(\eta)=\ominus\eta(\eta^\nu-c^\nu)^2(\eta^\nu-c_1^\nu)\cdot\dots\cdot(\eta^\nu-c_k^\nu)$, and $q(\eta)=\ominus\eta(\eta^\nu-c^\nu)^2(\eta^\nu-c_1^\nu)\cdot\dots\cdot(\eta^\nu-c_{k+l}^\nu)$ where $l\in\mathbb{N}$, $c,c_1,\dots,c_{k+l}\in\mathbb{C}^*$ and $c^\nu,c_1^\nu,\dots,c_{k+l}^\nu$ are pairwise different.

Consider (a). Assume $k\neq 0$. Then by Statement 8.2, $l=0$. From (b) of Proposition 9.3 we obtain

$$\frac{(k+2)\nu}{(k+1)\nu+1}=\frac{3+2n}{3s+3+n}=\frac{(2s+1)(2m+1)}{(2s+1)(m+2)}=\frac{2m+1}{m+2},$$

since from (d) of Proposition9.3 we obtain $n=(2s+1)m+s-1$ for some $m\in\mathbb{N}$.

The solutions of the above Diophantic equation with variables $k\in\mathbb{N}^*$, $m\in\mathbb{N}$ $\nu\in\mathbb{N}^*\setminus\{1\}$ are the following:

(A) $k=2$, $m=2$ and $\nu=5$. Therefore $n=5s+1$, $\deg(p)=20$ and $\deg(q)=16$;

(B) $k=1$, $m=3$ and $\nu=7$. Therefore $n=7s+2$, $\deg(p)=21$ and $\deg(q)=15$;

Consider (A). From (c) of Proposition 9.3 one has $D_{\mathcal{F}}=(10s+5)j=5(2s+1)j$.
$\deg(p_{\mathcal{F}})=i\deg(p)=20(2s+1)j$.
From (v) of Proposition 8.1 one has $M_{\mathcal{F}}=\gcd.(\deg(p),\deg(q))=4$.
From (d) of Proposition 9.3 one has $\kappa_{\mathcal{F}}(1-\pi(\mathcal{F}))=4$. Since $k=2$, one has $\lambda_{\mathcal{F}}\geq 2$. This gives (iii).

Consider (B). The same way as above, we have $D_{\mathcal{F}}=(14s+7)j$, $\deg(p_{\mathcal{F}})=21(2s+1)j$, $M_{\mathcal{F}}=3$ and $\kappa_{\mathcal{F}}(1-\pi(\mathcal{F}))=5$.

We may assume that $\mathcal{F}+c_1$ exists. Set $\mathcal{G}^*:=\mathcal{F}+c_1$. Assume $\mathcal{G}^*=I_Q(v^*)$ for some $Q\in\overline{R}_a\setminus R_a$ and $v^*\in\mathbb{Q}$. Since $\mathcal{G}^*\in T_a^{\nearrow}$, there exists $\mathcal{H}=I_Q(w)\in T_{a,\mathrm{cv}}$. Moreover, from Statement 9.3 we obtain $\lambda_{\mathcal{F}}\geq 2$. This gives (iv).

Consider the case $k=0$. Then $\deg(p)=2\nu$, $\deg(q)=(l+1)\nu+1$. Therefore, by (v) of Proposition 8.1 one has $M_{\mathcal{F}}=\gcd.(\deg(p),\deg(q))$ is 1 or 2. Moreover, $M_{\mathcal{F}}=2$ if and only if l is even and ν is odd. From (b) of Proposition 9.3 one has

$$\frac{2\nu}{(l+1)\nu+1}=\frac{3+2n}{3s+3+n}=\frac{(2s+1)(2m+1)}{(2s+1)(m+2)}=\frac{2m+1}{m+2},$$

since from (d) of Proposition9.3 we obtain $n = (2s+1)m + s - 1$ for some $m \in \mathbb{N}$.

The solution of this Diophantic equation with the assumption $\nu \in \mathbb{N}^* \setminus \{1\}$ takes the form: $l = 0$, $m = 3\phi + 1$, $\nu = 2\phi + 1$ for some $\phi \in \mathbb{N}^*$. Therefore $n = (2s+1)(3\phi+1) + s - 1$ From (c) of Proposition 9.3 one has $D_{\mathcal{F}} = (2s+1)(6\phi+3)j$.
$\deg(p_{\mathcal{F}}) = i \deg(p) = 2(2s+1)(2\phi+1)j$.
From (d) of Proposition 9.3 one has $\kappa_{\mathcal{F}}(1 - (\pi(\mathcal{F})) = 3\phi + 3$. Since the only c^* for which $\mathcal{F} + c^*$ exists, is c one gets $\lambda_{\mathcal{F}} = 0$. This gives (v).

Consider (b). Then by Statement 8.2 one has $l = 0$. In this case, by (v) of Proposition 8.1 one has $M_{\mathcal{F}} = \gcd.(\deg(p), \deg(q)) = \gcd.((k+2)\nu + 1, (k+1)\nu + 1) = 1$.

Consider the type (III) of Proposition 9.3. Then by Statements 3.16 and 8.2 one has $p(\eta) = \eta^2(\eta^\nu - c_1^\nu)\dots(\eta^\nu - c_k^\nu)$ and $q(\eta) = \eta(\eta^\nu - c_1^\nu)\dots(\eta^\nu - c_k^\nu)$ In this case, by (v) of Proposition 8.1 one has $M_{\mathcal{F}} = 1$.

Since there are no remaining cases, our proof is finished. $\square$.

Statement 9.12. *Let (f, g) be a normalized counterexample of the Jacobian conjecture. Assume that $T_{a,\text{pole}} = \{\mathcal{F}\}$. Then $\mathcal{F} \in FP_{\kappa,a,\text{pole}}$ cannot have type (3) of Proposition 9.1.*

Proof. Assume indirectly that $\mathcal{F}$ has type (3). Then $td(f, g) = 4$. Consider the characteristic sequence $\mathcal{F}_0, \dots, \mathcal{F}_n$.

From Statements 9.6, 9.7, 9.8, 9.9, 9.10 and 9.11we obtain that either the characteristic sequence has an element $\mathcal{F}_j$ with $M_{\mathcal{F}_j} = 1$, or (26) does not hold. The first case contradicts Proposition 8.4, the second contradicts Statement 9.4. $\square$

Theorem 9.1. *Assume that (f, g) is a counterexample of the Jacobian conjecture. Then $td(f, g) \geq 6$.*

Proof. Since the topological degree is invariant under composition with automorphisms, we may assume that (f, g) is a normalized.

By Propositions 9.1 and by Statement 9.12 one has $td(f, g) \geq 6$. $\square$

References

[A 'C] N. A 'Campo, *Sur la monodromie des singularités isolées d'hypersurfaces complexes*, Inv. Math. **20** (1973), 147-169.

[A] S. S. Abhyankar, *Expansion techniques in algebraic geometry*, Tata Inst. Fundamental Research, Bombay, 1977.

[A-M] S. S. Abhyankar and T.-T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 149-166.

[A-O] H. Appelgate and H. Onishi, *The Jacobian conjecture in two variables*, J. Pure Appl. Algebra, **37**, 215–227 (1985).

[B-C-W] H. Bass, E. Connell and D. Wright, *The Jacobian Conjecture: Reduction of degree and formal Expansion of the Inverse*, Bulletin of the AMS, **7** (1982), 287-330.

[B-E] M. de Bondt and A. van den Essen, *Nilpotent symmetric Jacobian matrices and the Jacobian Conjecture*, J. Pure Appl. Algebra, **193** (2004), no. 1-3, 61-70.

[B-E 2] M. de Bondt and A. van den Essen, *Nilpotent symmetric Jacobian matrices and the Jacobian Conjecture II*, Pure Appl. Algebra, **196** (2005), 135-148.

[B + M] S. Bochner and W. T. Martin, *Several complex variables*, Princeton Univ. Press, Princeton, N. J., 1948.

[C] L. A. Campbell, *A condition for a polynomial map to be invertible*, Math. Ann. **205** (1973), 243-248.

[D] A. V. Domrina, *On four-sheeted polynomial mappings of $\mathbb{C}^2$. The general case*, Mat. Zametki [Math. Notes], **65** (1999), no. 3, 464–467.

[D-O] A. V. Domrina and S. Yu. Orevkov, *On four-sheeted polynomial mappings of $\mathbb{C}^2$. I. The case of irreducible branch curve* Mat. Zametki [Math. Notes], **64** (1998), no. 6, 847–862.

[Egg] H. Eggers, *Polarinvarianten und die Topologie von Kurvesingularitäten*, Bonner Math Scrift **147** (1983)

[Ego] G. V. Egorov, *An example of a five-sheeted exotic covering over $\mathbb{C}^2$*, Mat. Zametki [Math. Notes], **71** (2002), no. 4, 532–547.

[Es] A. van den Essen, *Seven Lectures on Polynomial Automorphisms*, Univ. Nijmegen, Report No. 9442 (1994).

[Es2] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Univ. Nijmegen, Report No. 9540 (1995).

[Es 3] A. van den Essen *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Mathematics, **Vol. 190** Birkh"auser, 2000.

[G-M] R.V. Gurjar, M. Miyanishi, *On the Jacobian conjecture for $\mathbb{Q}$-homology planes*, J. Reine Angew. Math. **516** (1999) 115–132.

[He] R. C. Heitman, *On the Jacobian Conjecture*, J. Pure Appl. Algebra, **64** (1990) 35-72.

[Hu] A. Hurwitz, *Über Riemann'sche Flächen mit gegebene Verzweigungpunkten,* Math. Ann. **39** (1891) 1-60

[Ka] S. Kaliman, *On the Jacobian conjecture*, Proc. Amer. Math. Soc. **117** (1993), 45–51.

[Ke] O.H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. **47** (1939) 299–306.

[L] Lê Dũng Tráng, *The Jacobian Conjecture for rational polynomials* Act. Math. Vietnam. **32** (2007),295-301.

[L-W] Lê Dũng Tráng, C. Weber, *Polynomes a fibres rationnelles et conjecture jacobienne a 2 variables*, C. R. Acad. Sci. Paris **320** (1995) 581–584.

[M-L] G. Makar-Limanov, *Automorphisms of a free algebra on two generators*, Functional Anal. Appl. **4** (1970), 262-264 (Transi, from Funkcional Anal. i. Prilozen **4** (1970), 107-108.)

[M-M] M. Miyanishi, K. Masuda, *Generalized Jacobian conjecture and related topics* Proc. Internat. Conf. on Algebra, Arithmetic and Geometry, Tata Institute of Fundamental Research, 2000 427–466.

[M] T.-T. Moh, *On the global Jacobian conjecture and the configuration of roots*, J. Angew. Math., **340**, 140–212 (1983).

[N-N] W. D. Neumann and P. Norbury, *Nontrivial rational polynomials in two variables have reducible fibres* Bull. Austral. Math. Soc. **58** (1998), 501-503.

[N-S] A. Némethi and I. Sigray, *On the monodromy representation of polynomial maps in n variables*, Stud. Sci. Math Hung. **39** (2002), 361-367

[O] S. Yu. Orevkov, *Rudolph diagrams and an analytical realization of the Vitushkin covering*, Mat. Zametki [Math. Notes], **60** (1996), no. 2, 206–224.

[O 2] S. Yu. Orevkov, *On three-sheeted polynomial mappings of* $\mathbb{C}^2$ Izv. Akad. Nauk SSSR Ser. Mat. [Math. USSR-Izv.], **50** (1986), no. 6, 1231–1240.

[P] S. I. Pintchuk, *A Counterexample to the Strong Real Jacobian Conjecture* Math. Z. **217** (1994) 1-4.

[R] M. Razar, *Polynomial maps with constant jacobian* Israel J. Math. **32** (1979) 97-106.

[S] I. Sigray, *Solution of the polynomial equation* $kp'q - lpq' = cp^m$, to appear in Stud. Math. Sci. Hung.

[S-W] D. W. Sumners, J. M. Woods, *The monodromy of reducible plane curves*, Inv. Math. **40** (1977), 107-141.

[V] A. G. Vitushkin, *Some examples in connection with the problem of polynomial transformations of* $\mathbb{C}^n$, Izv. Akad. Nauk SSSR Ser. Mat. [Math. USSR-Izv.] **35** (1971), 269–279.

[V 2] A. G. Vitushkin, *Description of the homology of a branched covering over* $\mathbb{C}^2$, Mat. Zametki [Math. Notes], **64** (1998), no. 6

[Y] J. Yu *On the Jacobian conjecture: reduction of coefficients*, J. Algebra, **171**, 515–523 (1995).

10 Summary

Set $f, g : \mathbb{C}^2 \to \mathbb{C}$ polynomials. We say that (f, g) is a Jacobian pair if its Jacobian determinant is a non-zero constant. The plane Jacobian conjecture asserts that any Jacobian pair is an automorphism of $\mathbb{C}^2$.

For any polynomials $f, g : \mathbb{C}^2 \to \mathbb{C}$ one can find $d \in \mathbb{N}$, such that $\#(f, g)^{-1}(z) = d$ for generic $z \in \mathbb{C}^2$. This number $td(f, g) := d$ is called the topological degree of (f, g).

Orevkov and Domrina proved that if (f, g) is a counterexample of the Jacobian conjecture, then $td(f, g) \neq 3$ and $td(f, g) \neq 4$ cf. [O2], [D] and [D-O].

The main result of this thesis is:

Theorem 9.1 *Assume that (f, g) is a counterexample of the plane Jacobian conjecture. Then $td(f, g) \geq 6$.*

11 Összefoglalás

Legyenek $f, g : \mathbb{C}^2 \to \mathbb{C}$ polinomok. Azt mondjuk, hogy (f, g) egy Jacobi pár, ha Jacobi determinánsuk nem zéró konstans. A kétdimenziós Jacobi sejtés azt mondja ki, hogy bármely Jacobi pár $\mathbb{C}^2$ automorfizmusa.

Tetszőleges $f, g : \mathbb{C}^2 \to \mathbb{C}$ polinomhoz található olyan $d \in \mathbb{N}$, amelyre $\#(f, g)^{-1}(z) = d$ bármely generikus $z \in \mathbb{C}^2$ esetén. Az e módon definiált d számot az (f, g) topológiai fokának nevezzük.

Orevkov és Domrina bebizonyították, hogy amennyiben létezne a Jacobi sejtésre ellenpélda, akkor annak topológiai foka nem lehet 3 és nem lehet 4 ld. [O2], [D] és [D-O].

Az értekezés fő eredménye a 9.1 -es tétel, mely szerint amennyiben létezne a Jacobi sejtésre ellenpélda, akkor annak topológiai foka legalább 6 kell, hogy legyen.

Index